On critical submanifolds of the Willmore energy in four dimensions

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We establish a rigidity result for the critical points, with boundary, of the four dimensional Willmore energy (see [13] where this energy was studied from analytical standpoint). These critical points satisfy a 4-Willmore equation which is a sixth order nonlinear elliptic partial differential equation. We establish several curvature estimates and prove that four dimensional Willmore submanifold with totally geodesic boundary condition are umbilic.

The rigidity of several kinds of submanifolds has been widely studied in literature under different contexts. For instance, while some rigidity results for manifolds with bounded Ricci curvature were obtained in [2] other studies have focused on minimal submanifolds [3, 5, 6, 7, 11, 14, 16], critical points of the Willmore functional [8, 9, 10] and hypersurfaces of constant weighted mean curvature [1, 4, 15]. In [12], McCoy and Wheeler considered surfaces Σ immersed into $\mathbb{R}^3$ which are critical points of the functional

$$\int_\Sigma |\nabla H|^2 \, d\mu$$

and whose second fundamental form satisfies the smallness condition

$$\int_\Sigma |h|^2 \, d\mu \le \varepsilon$$

where ε is a small universal constant. They obtained the following result.

Theorem 1. *Let $f : \Sigma \to \mathbb{R}^3$ be an immersion satisfying*

$$\Delta^2 H + |h|^2 \Delta H - (h_0)^{ij} \nabla_i H \nabla_j H = 0$$

with the boundary conditions

$$|h| = 0 \quad \text{and} \quad \nabla_\eta H = \nabla_\eta \Delta H = 0.$$

If f also satisfies $\int_\Sigma |h|^2 d\mu \le \varepsilon$ for some sufficiently small $\varepsilon > 0$, then the immersed surface $f(\Sigma)$ is part of a flat plane, where η is the unit conormal to the boundary of Σ.

Our main result is the following rigidity theorem for critical points of the energy $\mathcal{E}(\Sigma)$.

Theorem 2. *Let $\vec{\Phi} : \Sigma \to \mathbb{R}^m$ be an immersion of a 4-dimensional manifold Σ satisfying $\int_\Sigma |\vec{h}|^2 d\mu \le \varepsilon$ and $\int_\Sigma |\vec{h}|^4 d\mu \le \varepsilon$ for some sufficiently small $\varepsilon > 0$. If $\vec{\Phi}$ also satisfies*

$$\vec{\mathcal{W}} = \vec{0} \tag{1}$$

together with the boundary conditions

$$\pi_{\vec{\eta}} \nabla \Delta_\perp \vec{H} = \pi_{\vec{\eta}} \nabla \vec{H} = \vec{0} \quad \text{and} \quad \vec{h} = \vec{0} \tag{2}$$

where

$$\begin{aligned}\vec{\mathcal{W}} := &-\frac{1}{2}\Delta_\perp^2 \vec{H} - \frac{1}{2}(\vec{h}_{ik} \cdot \Delta_\perp \vec{H})\vec{h}^{ik} - 4|\pi_{\vec{n}} \nabla \vec{H}|^2 \vec{H} + 2\pi_{\vec{n}} \nabla_j((\vec{h}_i^j \cdot \nabla^i \vec{H})\vec{H}) - 2\pi_{\vec{n}} \nabla_j((\vec{H} \cdot \vec{h}_i^j)\pi_{\vec{n}} \nabla^i \vec{H}) \\ &+ 2(\pi_{\vec{n}} \nabla_i \vec{H} \cdot \pi_{\vec{n}} \nabla_j \vec{H})\vec{h}^{ij} - \frac{1}{2}\Delta_\perp((\vec{H} \cdot \vec{h}^{ij})\vec{h}_{ij}) - 2\pi_{\vec{n}} \nabla_i \nabla_k((\vec{H} \cdot \vec{h}^{ik})\vec{H}) - 28|\vec{H}|^4 \vec{H} \\ &- \frac{1}{2}(\vec{H} \cdot \vec{h}^{ij})(\vec{h}_{ij} \cdot \vec{h}_{pq})\vec{h}^{pq} - 4(\vec{H} \cdot \vec{h}_{ij})(\vec{H} \cdot \vec{h}_k^i)\vec{h}^{jk} + 4|\vec{H} \cdot \vec{h}|^2 \vec{H} + 7\Delta_\perp(|\vec{H}|^2 \vec{H}) + 7|\vec{H}|^2(\vec{H} \cdot \vec{h}_{ij})\vec{h}^{ij}\end{aligned}$$

then the submanifold Σ is umbilic with totally geodesic boundary.

Note that the Willmore equation (1) is a sixth order nonlinear partial differential equation.

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