Fermat–Torricelli sets of finite sets of points in Euclidean plane

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Definition 1 ([8])**.** Let (X,ρ) be a metric space and $x_1,\ldots,x_n \in X$ be a finite collection of points in X. A point $\overline{x} \in X$ is called a *Fermat–Torricelli point* for $x_1,\ldots,x_n$ whenever for each $x \in X$ the following inequality holds true:

$$\sum_{k=1}^{n} \rho(\overline{x}, x_k) \le \sum_{k=1}^{n} \rho(x, x_k).$$

Definition 2. *Fermat–Torricelli set* for fixed points $\{x_1,\ldots,x_n\}$ is a set of all Fermat–Torricelli points for this collection of points.

In the case when $X = \mathbb{R}^n$ is the Euclidean space with the standard metric, then for every finite collection of points $x_1,\ldots,x_n \in \mathbb{R}^n$ the set of its Fermat–Torricelli points is non-empty, convex and compact. The problem of finding the Fermat–Torricelli set is called the *Fermat–Torricelli problem.*

This problem has both geometric and probabilistic interpretation. We can describe discrete probabilistic space $\Omega = \{x_1,\ldots,x_n\}$ with a probabilistic measure P on it, so that $\forall k \in \{1,\ldots,n\} : P(x_k) = \frac{1}{n}$. If for any $x_0 \in (X,\rho)$ we define a random variable $\xi_{x_0}(x) := \rho(x,x_0), x \in \Omega$, then Fermat–Torricelli set is the set of those $x_0 \in (X,\rho)$, for which random variable ξ_{x_0} has the least mathematical expectation.

Theorem 3. *Let A be the Fermat–Torricelli set for a collection $\{x_1,\ldots,x_n\}$, and B be the Fermat–Torricelli set for a collection $\{y_1,\ldots,y_k\}$ in Euclidean metric space $(\mathbb{R}^m,\rho)$ with standard metric. Assume that all points $x_1,\ldots,x_n,y_1,\ldots,y_k$ are mutually distinct. Then if $A\bigcap B \ne \varnothing$, then $A\bigcap B$ is the Fermat–Torricelli set for $\{x_1,\ldots,x_n,y_1,\ldots,y_k\}$.*

Since now we will name Euclidean metric space $(\mathbb{R}^2,\rho)$ with standard metric merely Euclidean plane.

Corollary 4. *If mutually distinct points x_1,x_2,x_3,x_4 in the Euclidean plane are vertices of a convex quadrilateral, then the point of intersection of its diagonals is a unique Fermat–Torricelli point for x_1,x_2,x_3,x_4.*

Corollary 5. *Let x_1,x_2,x_3,x_4 be mutually distinct points in the Euclidean plane laying on the same line in the given order. Then Fermat–Torricelli set of these points is the following set*

$$A = \{\alpha x_2 + (1-\alpha)x_3 \,|\, \alpha \in [0;1]\}.$$

Corollary 6. *Let x_1,x_2,x_3 be the vertices of some triangle in the Euclidean plane, and x_4 be some other point which lays on the side of triangle between x_2 and x_3. Then x_4 is a unique Fermat–Torricelli point of x_1,x_2,x_3,x_4.*

Corollary 7. *Let $A_1A_2\ldots A_n$ be a regular polygon with an even number of vertices. Then its center of gravity is a unique Fermat–Torricelli point of its vertices.*

Theorem 8. *Let $x_1,\ldots,x_n, n \ge 3$ be distinct points in the Euclidean plane. Then the following statements hold.*

1) *If* $x_1, \dots, x_n$ *lay on the same line in the given order and* n *is any even number, then the set*

$$A := \{x \in X \mid x = \alpha x_{\frac{n}{2}} + (1-\alpha)x_{\frac{n}{2}+1}, \alpha \in [0;1]\}$$

is a Fermat–Torricelli set for them.

2) *If* $x_1, \dots, x_n$ *lay on the same line in the given order and* n *is an odd number, then the point* $x_{\frac{n-1}{2}+1}$ *is a Fermat–Torricelli set for them.*

3) *If* $x_1, \dots, x_n$ *do not lay on the same line, then their Fermat–Torricelli point is unique.*

Lemma 9. *There is a unique point inside of triangle from which every side of triangle is visible under angle* 120° *if and only if every angle of this triangle is less than* 120°*.*

Different sources name this point in different ways: *Fermat point*, *Torricelli point*, and even *Steiner point* [8]. We will define it as *Steiner point for respective triangle.*

Theorem 10 ([8])**.** *Let* x_1, x_2, x_3 *be vertices of triangle in Euclidean plane every angle of which is less than* 120°*. Then the Steiner point for this triangle is a unique Fermat–Torricelli point for* x_1, x_2, x_3*.*

Theorem 11 ([8])**.** *Let* x_1, x_2, x_3 *be vertices of triangle in Euclidean plane one of whose angles is not less than* 120°*. Then the vertice, whose angle of triangle is not less than* 120°*, is a unique Fermat–Torricelli point of its vertices.*

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