Several forms of the geometric Lusternik-Schnirel'mann category

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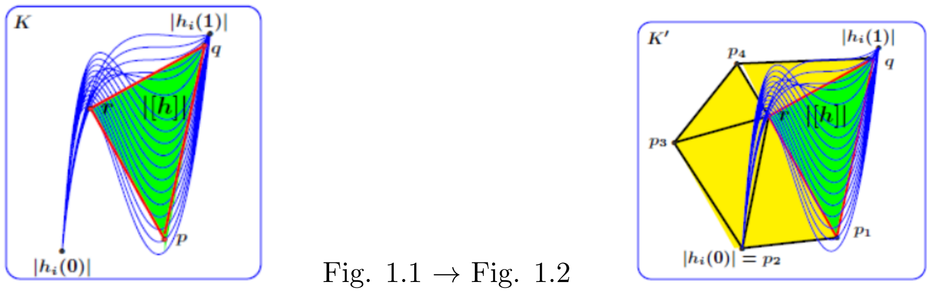

Fig. 1.1 → Fig. 1.2

Figure 1. Fig. 1.1 from Theorem 1.6 & Fig. 1.2 from Corollary 1.8.

This paper introduces results for several forms of the geometric Lusternik-Schirel'mann categories (LS gcat).

1. Geometric Lusternik-Schnirel'mann Category

Let $h : I \to K$ be a continuous map called **homotopy** (briefly, **path** in a space K. A **homotopic class** for different maps h (denoted by $[h] = \left\{h_0, \dots, h_i, \dots, h_{n-1[n]}\right\}$ with $[n] = \text{mod } n \in \mathbb{Z}^+$) is a collection of $h_{i[n]}$ homotopic maps that have the same endpoints, namely, $h_i(0)$ and $h_i(1)$. The geometric realization of $[n]$ (denoted by $|[h]|$) is a collection of sinusoidal curves, each being the geometric realization of a path h.

Definition 1.1 (Geometric LS Category)**.** [1] For a topological space X, the geometric category of X is the minimal covering of X with contractible open subsets of X.

Lemma 1.2. *Let $\boldsymbol{h}$ be a homotopic sinusoidal path. The geometric realization $|\boldsymbol{h}|$ is a planar sinusoidal curve.*

Lemma 1.3. *Let $[\boldsymbol{h}]$ be a collection of homotopic paths with common endpoints. The geometric realization $|[\boldsymbol{h}]|$ is a collection space filling planar curves.*

Lemma 1.4. *Let $\triangle \boldsymbol{pqr}$ be a planar filled triangle in a space K, geometric realization $|[\boldsymbol{h}]|$ such that each path h has endpoints $h(0), h(1) \in \boldsymbol{K} \setminus \triangle \boldsymbol{pqr}$. Then $\underset{i \to \infty}{limit}\, h_i \in [h] \supseteq \triangle \boldsymbol{pqr}$.*

Lemma 1.5. *Let h be a homotopic path in space K.*

1^o *Every path $\boldsymbol{h}$ is contractible.*
2^o *There exists a minimal $|[\boldsymbol{h}]|$ in space K with $h_i \in [h]$ with the same boundary endpoints $\boldsymbol{h(0), h(1)} \in \boldsymbol{\partial \triangle pqr}$ such that $|[\boldsymbol{h}]|$ covers triangle $\triangle \boldsymbol{pqr} \subset K$.*
3^o *Every planar triangle in K has a minimal covering $|[\boldsymbol{h}]|$.*

Theorem 1.6. *There exists $\boldsymbol{gcat}(|[\boldsymbol{h}]| \in \boldsymbol{2^K})$ such that $\boldsymbol{min}\, |[\boldsymbol{h}]| \supseteq \triangle \boldsymbol{pqr} \in \boldsymbol{2^K}$.*

[1] L. Montejano, Lusternik schirel'mann category: a geometric approach, Banach Cent. Publ. 18 (1986), 117–129.

Example 1.7. From Theorem 1.6, the triangle $\triangle \boldsymbol{pqr}$ in Fig. 1 has a $|[\boldsymbol{h}]|$ minimal covering, which is a $gcat(T)$.

A **cluster of triangles** $\{\triangle \boldsymbol{pqr}\}$ in a Euclidean space K is a collection of triangles attached to a common vertex. From Theorem 1.6, we have

Corollary 1.8. *Let* $|\{[\boldsymbol{h}]\}| \subset \boldsymbol{K}$ *such that each* $\boldsymbol{h} \in \{[\boldsymbol{h}]\} \subset \boldsymbol{K} \setminus \{\triangle \boldsymbol{pqr}\}$ *has the same endpoints. Then* $\exists \boldsymbol{gcat}(|\{[\boldsymbol{h}]\}|)$*:* $\boldsymbol{min}\,|\{[\boldsymbol{h}]\}| \supseteq \{\triangle \boldsymbol{pqr}\} \in \boldsymbol{2^{K}}$.

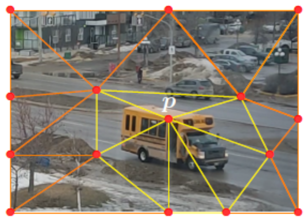

FIGURE 2. Delaunay Triangle cluster minimally covering bus in video frame foreground.

Example 1.9. From Corollary 1.8, there is a $\boldsymbol{gcat}(|\{[\boldsymbol{h}]\}|$ such that $\boldsymbol{gcat}(|\{[\boldsymbol{h}]\}|$ is a minimal covering of the triangle cluster $\{\triangle \boldsymbol{pqr}\} \cap \boldsymbol{p}$, which is a bounded region in Fig. 1.

2. Minimal Video Frame Foreground Object Covering

Delaunay triangulations[2] represent pieces of a continuous space in form of triangles with edges attached to selected vertices[3].

Theorem 2.1. *Let* $\mathcal{TC}$ *be a Delaunay triangle cluster with* k *triangles minimally covering a planar bounded region* $E \in 2^{\mathbb{R}^2}$*. Then* $\boldsymbol{gcat(\mathcal{TC}) = k}$.

Example 2.2. With restrictions on the selection of vertices (e.g., centroids), we obtain a minimal cluster $\mathcal{TC}$ of k triangles covering a bus, which is a bounded region in a Delauany triangulation of the video frame foreground in Fig. 2. Hence, from Theorem 2.1, $\boldsymbol{gcat(\mathcal{TC}) = k}$.

[2]B. Delaunay, Sur la sphère vide. a la mémoire de georges voronoï, Izvestia Akad. Nauk SSSR, Otdelenie Matematicheskii i Estestvennyka Nauk 7 (1934), 793–800.

[3]J.F. Peters, Proximal Voronoï regions, convex polygons, & Leader uniform topology, Advances in Math.: Sci. J. 4 (2015), no. 1, 1–5.