Fixed point theorem for mappings contracting perimeters of triangles and its generalizations

Evgeniy Petrov
(Institute of Applied Mathematics and Mechanics of the NAS of Ukraine, Slovyansk, Ukraine)
E-mail: eugeniy.petrov@gmail.com

Ruslan Salimov
(Institute of Mathematics of the NAS of Ukraine, Kiev, Ukraine)
E-mail: ruslan.salimov1@gmail.com

We establish two generalizations of the fixed point theorem for mappings contracting perimeters of triangles. In the first case we consider these mappings in semimetric spaces with triangle functions introduced by M. Bessenyei and Z. Páles. Such approach allows us to obtain corollaries for different types of semimetric spaces. In the second case we establish the fixed point theorem in ordinary metric spaces for more general class of mappings than mappings contractive perimeters of triangles.

Let X be a nonempty set. Recall that a mapping $d\colon X \times X \to \mathbb{R}^+$, $\mathbb{R}^+ = [0, \infty)$ is a *metric* if for all $x, y, z \in X$ the following axioms hold: (i) $(d(x, y) = 0) \Leftrightarrow (x = y)$; (ii) $d(x, y) = d(y, x)$; (iii) $d(x, y) \leqslant d(x, z) + d(z, y)$. The pair (X, d) is called a *metric space*. If only axioms (i) and (ii) hold then d is called a *semimetric*. A pair (X, d), where d is a semimetric on X, is called a *semimetric space*.

In 2017 M. Bessenyei and Z. Páles [1] introduced a definition of a triangle function $\Phi\colon \overline{\mathbb{R}}^2_+ \to \overline{\mathbb{R}}^+$ for a semimetric d. We use this definition in a slightly different form restricting the domain and the range of Φ by $\mathbb{R}^2_+$ and $\mathbb{R}^+$, respectively.

Definition 1. Consider a semimetric space (X, d). We say that $\Phi\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is a *triangle function* for d if Φ is symmetric and monotone increasing in both of its arguments, satisfies $\Phi(0, 0) = 0$ and, for all $x, y, z \in X$, the generalized triangle inequality

$$d(x, y) \leqslant \Phi(d(x, z), d(z, y))$$

holds.

Definition 2. Let (X, d) be a semimetric space with $|X| \geqslant 3$. We shall say that $T\colon X \to X$ is a *mapping contracting perimeters of triangles* on X if there exists $\alpha \in [0, 1)$ such that the inequality

$$d(Tx, Ty) + d(Ty, Tz) + d(Tx, Tz) \leqslant \alpha(d(x, y) + d(y, z) + d(x, z)) \tag{1}$$

holds for all three pairwise distinct points $x, y, z \in X$.

Note that the requirement for $x, y, z \in X$ to be pairwise distinct is essential. One can see that otherwise this definition is equivalent to the definition of contraction mapping.

Theorem 3. *Let (X, d), $|X| \geqslant 3$, be a complete semimetric space with the triangle function Φ satisfying the following three conditions:*

1) The inequality

$$\Phi(k\xi, k\eta) \leqslant k\Phi(\xi, \eta)$$

holds for all $k, \xi, \eta \in \mathbb{R}^+$.

2) For every $0 \leqslant \alpha < 1$ there exists $C(\alpha) > 0$ such that for every $p \in \mathbb{N}^+$ the inequality

$$\Phi(1, \Phi(\alpha, \Phi(\alpha^2, ..., \Phi(\alpha^{p-1}, \alpha^p)))) \leqslant C(\alpha)$$

holds.

3) Φ is continuous at $(0, 0)$.

Let the mapping $T\colon X \to X$ satisfy the following two conditions:

(i) $T(T(x)) \neq x$ for all $x \in X$ such that $Tx \neq x$.

(ii) T is a mapping contracting perimeters of triangles on X.

Then T has a fixed point. The number of fixed points is at most two.

Corollary 4. *Theorem 3 holds for semimetric spaces with power triangle functions $\Phi(x,y) = (x^q + y^q)^{\frac{1}{q}}$ if $q > 0$.*

If the usual triangle inequality is replaced by $d(x,y) \leqslant K(d(x,z) + d(z,y))$, $K \geqslant 1$, then (X,d) is called a *b-metric space*. The definition of a b-metric space was introduced by Czerwik [2].

Corollary 5. *Theorem 3 holds for b-metric spaces if $\alpha K < 1$, where α is the coefficient in (1).*

Definition 6. Let (X,d) be a metric space with $|X| \geqslant 3$ and let functions $F, G\colon \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ be such that for all $\xi, \eta, \zeta \in \mathbb{R}^+$ the following conditions hold:

$$F(\eta,\xi,\zeta) = F(\xi,\eta,\zeta) = F(\xi,\zeta,\eta),$$
$$G(\eta,\xi,\zeta) = G(\xi,\eta,\zeta) = G(\xi,\zeta,\eta),$$
$$G(\xi,\eta,\zeta) \geqslant \xi,$$
$$F(\xi,\eta,\zeta) \geqslant G(\xi,\eta,\zeta),$$
$$G(0,0,0) = 0 \text{ and } G \text{ is continuous at } (0,0,0).$$

We shall say that $T\colon X \to X$ is an (F,G)-*contracting mapping* on X if there exists $\alpha \in [0,1)$ such that the inequality

$$F(d(Tx,Ty), d(Ty,Tz), d(Tx,Tz)) \leqslant \alpha\, G(d(x,y), d(y,z), d(x,z))$$

holds for all three pairwise distinct points $x, y, z \in X$.

Theorem 7. *Let (X,d), $|X| \geqslant 3$, be a complete metric space and let $T\colon X \to X$ be a mapping satisfying the following two conditions:*

(i) $T(T(x)) \neq x$ for all $x \in X$ such that $Tx \neq x$.

(ii) T is an (F,G)-contracting mapping on X.

Then T has a fixed point. The number of fixed points is at most two.

If in Theorem 3 we set $\Phi(x,y) = x + y$ or in Theorem 7 we set $F(\xi,\eta,\zeta) = G(\xi,\eta,\zeta) = \xi + \eta + \zeta$, then we get the following.

Corollary 8. *Let (X,d), $|X| \geqslant 3$, be a complete metric space and let the mapping $T\colon X \to X$ satisfy the following two conditions:*

(i) $T(T(x)) \neq x$ for all $x \in X$ such that $Tx \neq x$.

(ii) T is a mapping contracting perimeters of triangles on X.

Then T has a fixed point. The number of fixed points is at most two.

References

[1] M. Bessenyei and Z. Páles. A contraction principle in semimetric spaces. *J. Nonlinear Convex Anal.*, 18(3): 515–524, 2017.

[2] S. Czerwik. Nonlinear set-valued contraction mappings in b-metric spaces. *Atti Semin. Mat. Fis. Univ. Modena*, 46(2): 263–276, 1998.