Structure of codimensional one flows on the 2-sphere with holes

Alexandr Prishlyak
(Taras Shevchenko National University of Kyiv)
E-mail: `prishlyak@yahoo.com`

First, we consider gradient vector fields on a sphere. Since the function increases along each trajectory, the field has no cycles and polycycles. In general position, a typical gradient field is a Morse field (Morse-Smale field without closed trajectories). In typical one-parameter families of gradient vector fields, two types of bifurcations are possible: saddle-node and saddle connection. The corresponding vector fields at the time of the bifurcation are fields of codimension one. In our case, they completely determine the topological type of the bifurcation. To classify Morse fields, a cell complex (diagram) is often used, in which cells of dimension n are stable manifolds of singular points with Morse index equal to n. We apply this approach to the classification of vector fields of codimension one.

Without loss of generality, we assume that under bifurcation (as the parameter increases), the number of singular points does not increase. The saddle-node bifurcation is defined by a pair of cells corresponding to the singular points participating in the bifurcation. We mark this pair on the diagram with a green arrow or a triangle. A saddle-node bifurcation in the diagram corresponds to a point of degree 3, where two edges (half-edges) are opposite and the third is perpendicular to them.

Then, the separatrix that connects the saddle with the node (source or sink) contracts to a point under the saddle-node bifurcation.

We describe all possible structures of Morse flows on S^2 with holes using separatrix diagrams and methods of papers [1, 2, 3, 4, 5].

Theorem 1. [6, 7] *The following types of gradient bifurcations are possible on spheres with holes:*
SN – internal saddle node; SC – internal saddle connection; BSN - boundary saddle node; BDS – boundary double saddle; HN – semi-boundary saddle node (node); HS – semi-boundary saddle-node (saddle); HSC – semi-boundary saddle connection; BSC – saddle connection of points on the boundary.

All possible structures of Morse flows and typical one-parameter bifurcations on spheres with holes in which no more than six singular points are given in Table 1.

Number of points	Morse	SN	SC	BSN	BDS	HN	HS	HSC	BSC
3 on D^2	2	0	0	0	0	2	0	0	0
4 on D^2	5	2	0	2	0	0	2	4	0
5 on D^2	7	8	0	2	0	6	8	4	0
6 on D^2	22	30	7	22	5	12	38	6	2
4 on $S^1 \times I$	2	0	0	0	0	0	0	0	1
5 on $S^1 \times I$	4	0	0	0	10	0	0	2	2
6 on $S^1 \times I$	14	4	2	14	6	4	18	10	9
6 on $F_{0,3}$	2	0	0	0	0	0	0	0	4

Table 1. Number of Morse flows and bifurcations on S^2 with holes (number of points befor bifurcation)

In what follows, we consider arbitrary, possibly non-gradient, flows on D^2. The optimal flow is the flow that has the least number of singular points among the flows of its type.

Theorem 2. *On a two-dimensional disk, there exist the following optimal codimensional one flow structures with degenerate singularities in the interior:*
SN: with a saddle knot - two (opposite);
HC: with a homoclonic cycle – two;
AN: Andronov-Hopf – two;
SL: with a saddle loop – two;
PC: with a parabolic cycle - two;
SC: with saddle ligament – six.
With singularities on the boundary, there exist the following optimal flows:
BSN: boundary saddle knot - two;
BHC: boundary saddle knot with a homoclinic boundary – two;
BDS: boundary double saddle – two;
BDSH: boundary double saddle with homoclinic boundary – one;
HN: semi-boundary saddle node (node) - two;
HS: semi-boundary saddle node (saddle) – four;
BDN: double nod on the boundary – two;
BDNH: double node with a homoclinic boundary – two;
HSC: semi-boundary saddle connection – two;
BSC: a connection of saddles on the boundary – three.
If the boundary is a parabolic cycle:
BPC: boundary parabolic cycle – two flow structures.

References

[1] Alexandr Prishlyak. Complete topological invariants of Morse-Smale flows and handle decompositions of 3-manifolds *Fundamentalnaya i Prikladnaya Matematika*, 11(4): 185–196, 2005.

[2] Alexandr Prishlyak, Andrii Prus, Morse-Smale flows on the torus with a hole *Proceedings of the International Geometry Cente*, 10(1): 47–58, 2017.

[3] Alexandr Prishlyak, Maria Loseva. Optimal Morse–Smale flows with singularities on the boundary of a surface *Journal of Mathematical Sciences*, 243: 279–286, 2019.

[4] Alexandr Prishlyak, Maria Loseva. Topology of Morse–Smale flows with singularities on the boundary of a two-dimensional disk *Proceedings of the International Geometry Center*, 9(2): 32–41, 2016.

[5] Alexandr Prishlyak. Topological equivalence of morse–smale vector fields with beh2 on three-dimensional manifolds *Ukrainian Mathematical Journal*, 54(4): 603–612, 2002.

[6] S. Bilun, B. Hladysh, A. Prishlyak, V. Sinitsyn. Gradient vector fields of codimension one on the 2-sphere with at most ten singular points *arXiv preprint arXiv:2303.10929*, 2023.

[7] S. Bilun, M. Loseva, O. Myshnova, A. Prishlyak. Typical one-parameter bifurcations of gradient flows with at most six singular points on the 2-sphere with holes *arXiv preprint arXiv:2303.14975*, 2023.