Convex bodies of constant width with exponential illumination number

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Borsuk's number $f(n)$ is the smallest integer such that any set of diameter 1 in the n-dimensional Euclidean space can be covered by $f(n)$ sets of smaller diameter. Currently best known asymptotic upper bound $f(n) \leq (\sqrt{3/2}+o(1))^n$ was obtained by Shramm (1988) and by Bourgain and Lindenstrauss (1989) using different approaches. Bourgain and Lindenstrauss estimated the minimal number $g(n)$ of open balls of diameter 1 needed to cover a set of diameter 1 and showed $1.0645^n \leq g(n) \leq (\sqrt{3/2}+o(1))^n$. On the other hand, Schramm used the connection $f(n) \leq h(n)$, where $h(n)$ is the illumination number of n-dimensional convex bodies of constant width, and showed $h(n) \leq (\sqrt{3/2}+o(1))^n$. The best known asymptotic lower bound on $h(n)$ is subexponential and is the same as for $f(n)$, namely $h(n) \geq f(n) \geq 1.2255^{\sqrt{n}}$ for large n established by Raigorodskii (1999). In 2015 Kalai asked if an exponential lower bound on $h(n)$ can be proved.

We show $h(n) \geq (\cos(\pi/14)+o(1))^{-n}$ by constructing the corresponding n-dimensional bodies of constant width, which answers Kalai's question in the affirmative. The construction is based on a geometric argument combined with a probabilistic lemma establishing the existence of a suitable covering of the unit sphere by equal spherical caps having sufficiently separated centers. The lemma also allows to improve the lower bound of Bourgain and Lindenstrauss to $g(n) \geq (2/\sqrt{3}+o(1))^n \approx 1.1547^n$.