Bifurcation points in random dynamical systems

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Let (M,ρ) be a locally compact separable metric space. By a continuous flows of mappings of M we will understand a family $(\theta_{s,t})_{-\infty<s\le t<\infty}$, such that

- for all $s\le t$ $\theta_{s,t}:M\to M$;
- for all $(s,x)\in\mathbb{R}\times M$ the mapping $t\mapsto\theta_{s,t}(x)$ is continuous and satisfies $\theta_{s,s}(x)=x$;
- for all $r\le s\le t$ $\theta_{s,t}\circ\theta_{r,s}=\theta_{r,t}$.

If $(\theta_{s,t})_{-\infty<s\le t<\infty}$ is a continuous flow of mappings of M and $\mathcal{D}=\{(s_n,x_n):n\ge 1\}$ is a countable dense set in $\mathbb{R}\times M$, then one can consider a sequence of continuous functions $\Phi_n(t)=\theta_{s_n,t}(x_n)$, $t\in[s_n,\infty)$, with the property

$$\max(s_n,s_m)\le s, \Phi_n(s)=\Phi_m(s)\Rightarrow\Phi_n|_{[s,\infty)}=\Phi_m|_{[s,\infty)} \qquad (1).$$

We are interested in the problem of recovering the flow $(\theta_{s,t})_{-\infty<s\le t<\infty}$ from the sequence of continuous functions $(\Phi_n)_{n\ge1}$, $\Phi_n\in C([s_n,\infty),M)$, that satisfy (1). Such problem naturally arises in the theory of stochastic flows. For example, if $\theta_{s,\cdot}(x)$ denotes the solution of the stochastic differential equation

$$dX(t)=a(X(t))dt+b(X(t))dW(t),\ X(s)=x, \qquad (2)$$

where W is a Brownian motion and a and b are continuously differentiable functions bounded together with their derivatives, then for all $r\le s\le t$ and $x\in M$, $\theta_{s,t}(\theta_{r,s}(x))=\theta_{r,t}(x)$ almost surely. However, the equality $\theta_{s,t}\circ\theta_{r,s}=\theta_{r,t}$ may not hold simultaneously for all $r\le s\le t$. This fact limits the possibility to apply the dynamic systems technique to the study of stochastic flows. The usual way to deal with this issue is to consider solutions of (2) for some dense sequence of initial values (s_n,x_n) and define solutions for other initial values by a limiting procedure. This strategy works well for stochastic flows of solutions to stochastic differential equations with sufficiently regular coefficients [1]. However, a lot of important stochastic flows are either generated by singular stochastic differential equations, or are not generated by stochastic differential equations at all [2]. This motivates the general question of a possibility to extend a sequence of continuous mappings $(\Phi_n)_{n\ge1}$ that satisfies (1) to a continuous flow $(\theta_{s,t})_{-\infty<s\le t<\infty}$ of mappings of M in the sense that $\Phi_n(t)=\theta_{s,t}(\Phi_n(s))$, $s_n\le s\le t$.

Our main result is the following. Assume that $(\Phi_n)_{n\ge1}$ is a sequence of continuous mappings, $\Phi_n\in C([s_n,\infty),M)$, that satisfies (1) and is such that the sequence $((s_n,\Phi_n(s_n)))_{n\ge1}$ is dense in $\mathbb{R}\times M$, and for every compact $L\subset\mathbb{R}\times M$ the set

$$\left\{\Phi_n|_{[s,\infty)}:s_n\le s,(s,\Phi_n(s))\in L\right\}$$

is relatively compact with respect to the topology of uniform convergence on bounded intervarls. Consider sets $\mathcal{K}^{s,t}_x=\bigcap_{\varepsilon>0}\overline{\{\Phi_n|_{[s,t]}:s_n\le s,\rho(\Phi_n(s),x)\le\varepsilon\}}$, and let

$$E=\{(s,x)\in\mathbb{R}\times M:\forall t>s\ \mathcal{K}^{s,t}_x \text{ contains at least two distinct functions}\}.$$

Assume that F is a closed subset of $\mathbb{R}\times M$, such that $E\subset F$.

Theorem 1. *Let $(\theta_{s,t}:-\infty<s\le t<\infty)$ be a family of mappings of M. Define*

$$\sigma^s_x=\inf\{t>s:\theta_{s,t}(x)\in F\}.$$

Assume that

- *for all $t\in(s,\sigma^s_x)$, $\theta_{s,t}(x)\in\mathcal{K}^{s,t}_x$;*
- *if $s_n\le s\le t$, then $\theta_{s,t}(\Phi_n(s))=\Phi(t)$;*
- *if $\sigma^s_x\le t$, then $\theta_{\sigma^s_x,t}(\theta_{s,\sigma^s_x})=\theta_{s,t}(x)$;*

- *if* $t > \sigma^s_x$, *and* $\theta_{s,t}(x) \in F$, *then there exists* $n \geq 1$, *such that* $s_n \leq t$ *and* $\theta_{s,\cdot}(x)|_{[t,\infty)} = \Phi_n|_{[t,\infty)}$.

Then for all $r \leq s \leq t$ $\theta_{s,t} \circ \theta_{r,s} = \theta_{r,t}$.

We will give applications of the theorem to analogues of Arratia and Burdzy-Kaspi flows on metric graphs.

References

[1] Hiroshi Kunita. *Stochastic flows and stochastic differential equations*, volume 24 of *Cambridge studies in advanced mathematics*. Cambridge University Press, 1990.

[2] Yves Le Jan, Olivier Raimond. *Flows, Coalescence and Noise*, 32(2) : 1247–1315, 2004.