On equicontinuity of families of mappings with one normalization condition by the prime ends

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A Borel function $\rho : \mathbb{R}^n \rightarrow [0,\infty]$ is called *admissible* for Γ, abbr. $\rho \in \operatorname{adm}\Gamma$, if $\int\limits_{\gamma} \rho(x)|dx| \geqslant 1$ for each (locally rectifiable) $\gamma \in \Gamma$. We define the quantity

$$M(\Gamma) = \inf_{\rho \in \operatorname{adm}\Gamma} \int\limits_{\mathbb{R}^n} \rho^n(x)\, dm(x) \tag{1}$$

and call $M(\Gamma)$ *a modulus* of Γ; here m stands for the n-dimensional Lebesque measure, see [1, 6.1].

Given sets E and F and a domain D in $\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$, we denote $\Gamma(E,F,D)$ the family of all paths $\gamma : [0,1] \rightarrow \overline{\mathbb{R}^n}$ joining E and F in D, that is, $\gamma(0) \in E$, $\gamma(1) \in F$ and $\gamma(t) \in D$ for all $t \in [0,1]$.

An *end* of a domain D is an equivalence class of chains of cross-cuts of D. We say that an end K is a *prime end* if K contains a chain of cross-cuts $\{\sigma_m\}$, such that

$$\lim_{m\rightarrow\infty} M(\Gamma(C,\sigma_m,D)) = 0$$

for some continuum C in D. Set $\mathbb{B}^n := \{x \in \mathbb{R}^n : |x| < 1\}$. We say that the boundary of a domain D in $\mathbb{R}^n$ is *locally quasiconformal* if every point $x_0 \in \partial D$ has a neighborhood U that admit a conformal mapping φ onto the unit ball $\mathbb{B}^n \subset \mathbb{R}^n$ such that $\varphi(\partial D \cap U)$ is the intersection of $\mathbb{B}^n$ and a coordinate hyperplane, see e.g. [2], cf. [3]. We say that a bounded domain D in $\mathbb{R}^n$ is *regular* if D may be mapped quasiconformally onto a bounded domain with a locally quasiconformal boundary. If $\overline{D}_P$ is the completion of a regular domain D by its prime ends and g_0 is a quasiconformal mapping of a domain D_0 with locally quasiconformal boundary onto D, then this mapping naturally determines the metric $\rho_0(p_1,p_2) = \left|\widetilde{g_0}^{-1}(p_1) - \widetilde{g_0}^{-1}(p_2)\right|$, where $\widetilde{g_0}$ is the extension of g_0 onto $\overline{D_0}$. Let $x_0 \in \overline{D}$, $x_0 \neq \infty$, $S(x_0,r) = \{x \in \mathbb{R}^n : |x - x_0| = r\}$, $A = A(x_0,r_1,r_2) = \{x \in \mathbb{R}^n : r_1 < |x - x_0| < r_2\}$.

Let $f : D \rightarrow \mathbb{R}^n$, $n \geqslant 2$, and let $Q : \mathbb{R}^n \rightarrow [0,\infty]$ be a Lebesgue measurable function such that $Q(y) \equiv 0$ for $y \in \mathbb{R}^n \setminus f(D)$. Let $A = A(y_0,r_1,r_2)$ and let $\Gamma_f(y_0,r_1,r_2)$ denotes the family of all paths $\gamma : [a,b] \rightarrow D$ such that

$$f(\gamma) \in \Gamma(S(y_0,r_1), S(y_0,r_2), A(y_0,r_1,r_2))\,,$$

i.e., $f(\gamma(a)) \in S(y_0,r_1)$, $f(\gamma(b)) \in S(y_0,r_2)$, and $f(\gamma(t)) \in A(y_0,r_1,r_2)$ for any $a < t < b$.

We say that, f *satisfies the inverse Poletsky inequality* at a point $y_0 \in f(D)$, if the relation

$$M(\Gamma_f(y_0,r_1,r_2)) \leqslant \int\limits_{f(D)\cap A(y_0,r_1,r_2)} Q(y) \cdot \eta^n(|y - y_0|)\, dm(y) \tag{2}$$

holds for any Lebesgue measurable function $\eta : (r_1,r_2) \rightarrow [0,\infty]$ satisfying the relation

$$\int\limits_{r_1}^{r_2} \eta(r)\, dr \geqslant 1\,. \tag{3}$$

We say that the boundary of D is *weakly flat* at a point $x_0 \in \partial D$ if, for every number $P > 0$ and every neighborhood U of the point x_0, there is a neighborhood $V \subset U$ such that $M(\Gamma(E, F, D)) \geqslant P$ for all continua E and F in D intersecting ∂U and ∂V. We say that the boundary ∂D is weakly flat if the corresponding property holds at every point of the boundary.

Given domains $D, D' \subset \mathbb{R}^n$, $n \geqslant 2$, points $a \in D$, $b \in D'$ and a Lebesgue measurable function $Q : D' \to [0, \infty]$ denote $\mathfrak{S}_{a,b,Q}(D, D')$ a family of all open discrete and closed mappings f of D onto D', satisfying the relation (2) for any $y_0 \in D'$, while $f(a) = b$. The following statement holds.

Theorem 1. *Assume that, D has a weakly flat boundary, any component of which does not degenerate into a point. If $Q \in L^1(D')$ and D' is regular, then any $f \in \mathfrak{S}_{a,b,Q}(D, D')$ has a continuous extension $\overline{f} : \overline{D} \to \overline{D'}_P$, $\overline{f}(\overline{D}) = \overline{D'}_P$, and, in addition, a family $\mathfrak{S}_{a,b,Q}(\overline{D}, \overline{D'})$ which consists of all extended mappings $\overline{f} : \overline{D} \to \overline{D'}_P$, is equicontinuous in $\overline{D}$.*

The result mentioned above is published in [4].

References

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