Equiaffine immersions of codimension two with flat connection

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We consider the affine immersions by K. Nomizu, T. Sasaki [1], namely $f : (M^n, \nabla) \to \mathbb{R}^{n+2}$. For a transversal frame $\xi_1,\ \xi_2$ and tangent vector fields $X,\ Y$ we have the affine analogues of Gauss and Weingarten decompositions, namely

$$D_X f_*(Y) = f_*(\nabla_X Y) + h^\alpha(X,\ Y)\xi_\alpha,$$
$$D_X \xi_\alpha = -f_*(S_\alpha X) + \tau_\alpha^\beta(X)\xi_\beta,$$

where h^α are components of the affine fundamental form, S_α are shape operators, τ_α^β are forms of transversal connection (with respect to $\xi_1,\ \xi_2$).

The *Weingarten mapping* $S_x : Q_x \times T_x(M^n) \to T_x(M^n)$ is defined [2] as follows: $(\xi, X) \mapsto S_\xi X$ at every point $x \in M^n$ (where $T_x(M^n)$ and Q_x are tangent and transversal distributions.)

For an affine immersion $f : (M^n, \nabla) \to \mathbb{R}^{n+2}$ with a transversal frame $\{\xi_1,\ \xi_2\}$, an *induced volume element* θ on M^n is defined [1, 3, 4] as follows:

$$\theta(X_1, \dots, X_n) = \det(f_*(X_1), \dots, f_*(X_n), \xi_1, \xi_2).$$

The transversal distribution Q with frame $\{\xi_1,\ \xi_2\}$ is called *equiaffine* if $\nabla_X \theta = 0$ for all $X \in T_x(M^n), x \in M^n$. For two-codimension affine immersion this condition is equivalent [4] to

$$\tau_1^1(X) + \tau_2^2(X) \equiv 0.$$

With an equiaffine transversal distribution Q we have an *equiaffine structure* $(\nabla,\ \theta)$ on M^n.

We will consider an affine immersion $f : (M^n, \nabla) \to \mathbb{R}^{n+2}$ with flat connection ∇ and equiaffine transversal distribution. Two-codimensional affine surfaces with different additional properties have been studied by many authors. Flat affine surfaces in $\mathbb{R}^4$ with flat normal connection were studied in [3]. The description of a parallel affine immersions $(M^n, \nabla) \to \mathbb{R}^{n+2}$ with flat connection in dependence on the rank of the Weingarten mapping were given in [2].

Let us remind that in general case (codimension k) the kernel and the image of the Weingarten mapping is defined by $\ker S = \bigcap\limits_{\alpha=1}^{k} \ker S_\alpha,\ \text{im}\ S = \bigcup\limits_{\alpha=1}^{k} \text{im}\ S_\alpha$. We say that Weingarten mapping is p-dimensional if $\text{rank} S := \dim \text{im} S = p$. It was proved [6] that for the immersion $f : (M^n, \nabla) \to \mathbb{R}^{n+k}$ (for $k < n$) with maximal pointwise codimension and flat connection ∇ the following relations hold true:

$$1)\ \dim\ker S \geq n-k; \qquad 2)\ \ker h \subseteq \ker S; \qquad 3)\ \dim\text{im}\ S \leq k;$$

$$4)\ \text{if}\ \ \dim\text{im}\ S = k,\ \text{ then }\ \dim\ker S = n-k \text{ and } \ker h = \ker S.$$

It was also proved [6] that the distribution $\mathcal{S}$ of the kernels of Weingarten mapping is integrable on M^n and there exists a transversal distribution which is stationary along the leaves of the foliation $\mathcal{FS}$.

Since in the case of codimension two we have $\dim\text{im}\ S \leq 2$, $\dim\ker S \geq n-2$, so we have only three possible values for the dimension of im S, namely $0,\ 1,\ 2$. The most studied are affine immersions with zero and two-dimensional Weingarten mapping.

It is well known that an affine immersion with zero Weingarten mapping ($S \equiv 0$) has a flat connection and it is affinely equivalent to the graph of certain smooth map $F : M^n \to \mathbb{R}^2$ (see for example [5, 1, 6]), i. e.

$$f : (u^1, \dots,\ u^n) \mapsto (u^1, \dots,\ u^n,\ f^1(u^1, \dots,\ u^n),\ f^2(u^1, \dots,\ u^n)).$$

Obviously, a graph immersion is equiaffine.

According to the properties which were discussed in [6], in case $\dim \operatorname{im} S = 2$ we obtain $\ker h = \ker S$ and $\dim \ker h = n - 2$. Therefore such a submanifold is a submanifold of rank two (by the rank of Gaussian (Grassmann) mapping) [7]. Rank-two submanifold is a ruled submanifold with $(n-2)$-dimensional rulings over a two-dimensional base. In case this submanifold is a cylinder, its connection is determined by the connection of the cylinder base. In the general case the problem on its connection remains open.

We obtain a parametrization of a submanifold with one-dimensional Weingarten mapping and given properties. Such a submanifold is a peculiar "mix" of a graph and a ruled submanifold.

The main result. *Let $f : (M^n, \nabla) \to \mathbb{R}^{n+2}$ be an affine immersion with rank two affine fundamental form, equiaffine structure, flat connection ∇, one-dimensional Weingarten then there exists three types of its parametrization:*

(i) $\vec{r} = g(u^1, \ldots, u^n)\vec{a}_1 + \int \vec{\varphi}(u^1)du^1 + \sum\limits_{i=2}^{n} u^i \vec{a}_i;$

(ii) $\vec{r} = (g(u^2, \ldots, u^n) + u^1)\vec{a} + \int v(u^1)\vec{\eta}(u^1)du^1 + \sum\limits_{i=2}^{n} u^i \int \lambda_i(u^1)\vec{\eta}(u^1)du^1;$

(iii) $\vec{r} = (g(u^2, \ldots, u^n) + u^1)\vec{\rho}(u^1) + \int (v(u^1) - u^1)\dfrac{d\vec{\rho}(u^1)}{du^1}du^1 + \sum\limits_{i=2}^{n} u^i \int \lambda_i(u^1)\dfrac{d\vec{\rho}(u^1)}{du^1}du^1.$

References

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