Investigation of the connection between different models of topologies on a finite set

Anna Skryabina
(Department of Universal Mathematics, Zaporizhzhya National University, Zhukovskogo str. 66, building 1, office 21-A, Zaporizhzhya, 69600, Ukraine)
E-mail: anna_29_95@ukr.net

Polina Stegantseva
(Department of Universal Mathematics, Zaporizhzhya National University. Zhukovskogo str. 66, building 1, office 21-A, Zaporizhzhya, 69600, Ukraine)
E-mail: stegpol@gmail.com

One of the unsolved problems of discrete mathematics is the problem of counting all topologies on a finite set. Topologies on a finite set were modeled using various mathematical objects (graphs, partial order relations, Boolean functions and their normal forms, $(0,1)$-matrices of a special form, etc.). In [1] topologies were studied using the topology vector, the concept of which was introduced in [2]. In [3], in addition to the topology vector, Boolean functions and a maximal 2-CNF were used. The question arises about the relationship between the objects of different models, which can be used both to continue research and to verify the results. This issue was partially raised by us in our work [5].

In this paper, we consider the connection between models in the form of $(0,1)$-matrices of a special form and in the form of ordered sets $(M_1, M_2, ..., M_n)$ of minimal neighborhoods of elements of a given ordered finite set $X = (x_1, x_2, ..., x_n)$ (using such sets, one can instantly pass to topology vectors - ordered sets of integers α_k, which were effectively used in [1,2]).

According to [4] the $(0,1)$-matrix σ_{ij}, where $1 \leq i, j \leq n$, corresponds to some topology on X (in this case, this matrix is called a grid, and its order is the order of the grid) if and only if the following conditions are true:

1) $\sigma_{ij} = 1$, if $x_i \in \overline{x}_j$,
2) $\sigma_{ij} = 0$ in the other case

(Here the symbol $\overline{x}_j$ indicates the closure of a point x_j in a given topology).

Let, in the i-line of the matrix $\sigma_{ir_1} = 1$,$\sigma_{ir_2} = 1$,...,$\sigma_{ir_k} = 1$, the other elements be equal to zero. From $\sigma_{ir_1} = 1$ follows $x_i \in \overline{x}_{r_1}$, and then $M_i \cap \{x_{r_1}\} \neq \varnothing$. So, $x_{r_1} \in M_i$. Similarly, with $\sigma_{ir_2} = 1$ we get $x_{r_2} \in M_i$ etc. Hence, $M_i \supseteq \{x_i, x_{r_1}, ..., x_{r_k}\}$. On the other hand, for an element x_p from M_i the inclusion $x_i \in \overline{x}_p$ is obvious. Then $\sigma_{ip} = 1$. Thus, the i-line of the $(0,1)$-matrix corresponds to the minimal neighborhood of the element x_i.

The connection found made it possible to prove some properties of networks using minimal neighborhoods. In particular, the following properties of networks:

1) $\sigma_{ii} = 1$ at all $i = 1, ..., n$;
2) if $\sigma_{ir} = 1$ and $\sigma_{rj} = 1$, then $\sigma_{ij} = 1$;
3) The network σ defines T_0-topology on X (is T_0-network) if and only if $\sigma_{ij}\sigma_{ji} = 0$ at $i \neq j$.

Using these and other properties of networks and their connection with sets of minimal neighborhoods (bases of topologies), we enumerate all possible networks of T_0-topologies on a 4-element set and find the total number of T_0-topologies and the number 355 all of the topologies on this set using the well-known formula from work [6].

References

[1] Stegantseva P.G., Skryabina A.V. Topologies on the n-element set consistent with topologies close to the discrete on an $(n-1)$-element set. *Ukrainian Mathematical Journal*, No. 2. Vol. 73. : 276-288, 2021.

[2] Velichko I. G., Stegantseva P. G., Bashova N. P. Perechislenie topologii blizkikh k diskretnoi na konechnikh mnozhestvakh. *Izvestiya vuzov. Matematika*, No. 11 : 23-31, 2015.

[3] Skryabina Anna, Stegantseva Polina, Bashova Nadia. The properties of 2-CNF of the mutually dual and self-dual T_0-topologies on the finite set and the calculation of T_0-topologies of a certain weight. *Proceedings of the International Geometry Center*, No. 1. Vol. 15. : 75-85, 2022.

[4] Borevich Z. I. K voprosu perechisleniya konechnikh topologii. *Zap. nauch. sem. LOMI. Leningrad: Nauka*, Vol. 71. : 47-65, 1977.

[5] Styeganceva P.G., Skryabina A.V. Zastosuvannya riznih modelej topologij na skinchennih mnozhinah dlya doslidzhennya yih kilkosti ta strukturi. *Modern aspects of science. 22- th volume of the international collective monograph.* : 481-505, 2022.

[6] Evans J. W., Harary F., Lynn M. S. On the computer enumeration of finite topologies. *Communications of the ACM.* No. 5. Vol. 10. : 295-297, 1967.