In this research we continue our previous investigation of wreath product normal structure [1].

The lattice of normal subgroups and their properties for finite iterated wreath products $S_n \wr \ldots \wr S_n$, $n, m \in \mathbb{N}$ are found. Special classes of normal subgroups and their orders and generators are found. Further, the monolith of these wreath products is investigated by us.

Let $k(\pi)$ be the number of cycles in decomposition of permutation $\pi$ of degree $n$.

The number $n - k(\pi)$ is denoted by $\text{dec}(\pi)$, and is called a decrement [2] of permutation $\pi$.

As well known [2] the minimal number of transpositions in factorization of a permutation $\pi$ on transpositions is happen to be equal to $\text{dec}(\pi)$. We set $\text{dec}(e) = 0$. Therefore the decrement of $n$-cycle is $n - 1$.

If $\pi_1, \pi_2 \in S_n$, then the following formula holds:

$$\text{dec}(\pi_1 \cdot \pi_2) = \text{dec}(\pi_1) + \text{dec}(\pi_2) - 2m, m \in \mathbb{N},$$

where $m$ is number of joint simplifying transpositions in $\pi_1$ and $\pi_2$.

The trivial subgroup of $S_n$ we denote by $E$.

**Definition 1.** The set of elements from $S_n \wr S_n$, $n \geq 5$ or $n = 3$ of the tableaux form: $[e]_1, [a_1, a_2, \ldots, a_n]_2$, satisfying the following condition

$$\sum_{i=1}^{n} \text{dec}([a_i]_2) = 2k, k \in \mathbb{N},$$

we will call set of type $\vec{A}^{(2)}$ and denote this set by $E \wr \vec{A}_n$. For brevity of notation this subgroup be also denoted by $\vec{A}^{(2)}_n$. It follows directly from the definition that the set of these elements supplemented by the operation of multiplication in the subdirect product, coincides with the group $E \times (S_n \boxtimes S_n \boxtimes S_n \ldots \boxtimes S_n)$, where subdirect product satisfies to condition (2).

We remind that the intersection of all non-trivial normal subgroups $\text{Mon}(G)$ of $G$ is called the monolith of a group $G$.

**Proposition 2.** Elements of first type form the subgroup $e \wr A_n$. This subgroup is the monolith of $S_n \wr S_n$.

Now we can recursively construct easiest and elegant subgroup $E \wr \vec{A}^{(2)}_n$ of $S_n \wr S_n \wr S_n$.

**Definition 3.** The subgroup $E \wr \vec{A}^{(2)}_n$ be denoted by $\vec{A}^{(3)}_n$.

The order of $E \wr \vec{A}^{(2)}_n$ is $(n!)^3 : 2^3$. Furthermore we prove that $E \wr \vec{A}^{(2)}_n \lhd S_n \wr S_n \wr S_n$.

Let the set of elements from $S_n \wr S_n \wr S_n$, $n \geq 3$ of the form:

$$[e]_1, [e, e, \ldots, e]_2, [a_1, a_2, \ldots, a_n]_3$$

satisfying the following condition
\[
\sum_{i=1}^{n^2} \text{dec}([a_i]_3) = 2k, \ k \in \mathbb{N},
\]
be denoted by \(\overline{A}_{n^2}^{(3)}\).

**Proposition 4.** The set of elements of type \(\overline{A}_{n^2}^{(3)}\) forms a subgroup in \(S_n \wr S_n \wr S_n\). Moreover \(\overline{A}_{n^2}^{(3)} \triangleleft S_n \wr S_n \wr S_n\).

**Remark 5.** We note that \(\overline{A}_{n^2}^{(3)} < \overline{A}_{n^2}^{(3)}\). The order of \(\overline{A}_{n^2}^{(3)}\) is \((n!)^n : 2\). Furthermore \(\overline{A}_{n^2}^{(3)} \triangleleft S_n \wr S_n \wr S_n\).

**Definition 6.** A subgroup in \(S_n \wr S_n \wr S_n\) is called \(\overline{T}_n\) if it consists of:

1) elements of \(E \wr A_n\),
2) elements with the tableau \(\overline{3}\) presentation \([e]_1, [\pi_1, \ldots, \pi_n]_2\), that \(\pi_i \in S_n \setminus \overline{A}_n\).

One easy can validates a correctness of this definition, i.e. that the set of such elements form a subgroup and its normality. This subgroup has structure \(\overline{T}_n \simeq (A_n \times A_n \times \cdots \times A_n) \rtimes C_2 \simeq S_n \wr S_n \cdots \wr S_n\), where the operation of a subdirect product \(\boxplus\) is subject to items 1) and 2).

**Definition 7.** A subgroup in \(S_n \wr S_n \wr S_n\) is of the type \(\overline{T}_n^{(3)}\) if it consists of:

1) elements of the form \(E \wr E \wr A_n\),
2) elements with the tableau \(\overline{3}\) presentation \([e]_1, [e, \ldots, e]_2, [\pi_1, \ldots, \pi_n, \pi_{n+1}, \ldots, \pi_n]_3\), wherein \(\forall i = 1, \ldots, n: \pi_i \in S_n \setminus \overline{A}_n\).

We define recursively the subgroup \(\overline{T}_n^{(3)}\) having \(n\) different intervals of elements with the same parity permutations on \(X^2\).

**Definition 8.** The subgroup of \(S_n \wr S_n \wr S_n\) having structure \(E \wr \overline{T}_n\) is denoted by \(\overline{T}_n^{(3)}\). The following isomorphism \(\overline{T}_n^{(3)} \simeq S_n \bigoplus S_n \cdots \bigoplus S_n \times S_n \bigoplus S_n \cdots \bigoplus S_n \times \cdots \times S_n \bigoplus S_n \cdots \bigoplus S_n\), where a tuple \(S_n \bigoplus S_n \cdots \bigoplus S_n\) repeats \(n\) times, holds. The operation of a subdirect product \(\boxplus\) is determined by Definition 1.

The operation \(\boxplus\) accords with the properties described in item 1 and 2 of Definition 6. Also \(\boxplus\) is determined by automorphism in \(\overline{T}_n \simeq (A_n \times A_n \times \cdots \times A_n) \rtimes C_2\) in this case.

**Remark 9.** Note that in \(\overline{T}_n^{(3)}\) vertex permutation of tableau third part satisfy the condition: elements with the tableau presentation \([e]_1, [e, \ldots, e]_2, [\pi_1, \ldots, \pi_n; \pi_{n+1}, \ldots, \pi_n]_3\), that either all \(\pi_i \in S_n \setminus \overline{A}_n\) or all \([\pi_i]_3 \in A_n\) for \(1 \leq i \leq n\), \(n + 1 \leq i < 2n\), \(\ldots\), \(n^2 - n \leq i \leq n^2\).

Here are the names of (almost all) predefined theorem-like environments.

**Proposition 10.** The subgroup \(E \wr A_n\) is the monolith of \(S_n \wr S_n\).

We call level of \(AutX^*\) as active if it has at least one non-trivial permutation. Denote by \(Aut_f X^*\) the group of all finite automorphism of spherically homogeneous rooted tree.

**Proposition 11.** Let \(H \triangleleft Aut_f X^*\) with depth \(k\) then \(H\) contains \(k\)-th level subgroup \(P\) having all even vertex permutations \(p_{ki} \in A_n\) on \(X_k\) and trivial permutations in vertices of rest of levels. Furthermore \(P\) is normal in \(W\) provided \(k\) is last active level of \(Aut_f X^*\).
Theorem 12. Proper normal subgroups in $S_n \wr S_m$ (action of group is left), where $n, m \geq 3$ with $n, m \neq 4$ are of the following types:

1) the subgroups of the first level stabilizer are $E \wr A_m$, $E \wr S_n$, $E \wr A_n$.

2) the subgroups that act on both levels are $A_n \wr \tilde{A}_m$, $S_n \wr \tilde{A}_m$, $A_n \wr S_m$.

In total there are 8 proper normal subgroups in $S_n \wr S_m$.

Proposition 13. All normal subgroups of $S_n \wr (S_m \times S_k)$ can be partitioned in 2 types:

1. $E \wr (N_i \times N_j)$, where $N_i \leq \prod_{k=1}^n S_m^{(k)}$ and $N_j \leq \prod_{l=1}^n S_l^{(l)}$.
2. $\tilde{A}_i \wr (N_i \times N_j)$, where $\tilde{A}_i \leq S_n$, $N_i$ and $N_j$ are subgroups from item 1) possessing an extension by $\tilde{A}_i$ in a correspondent groups $S_n \wr S_m$ and in $S_n \wr S_k$. The full list of them: $S_n \wr (S_m \times \tilde{A}_k)$, $S_n \wr (\tilde{A}_m \times S_k)$, also $A_n \wr (S_m \times \tilde{A}_k)$, $A_n \wr (\tilde{A}_m \times S_k)$. We denote the set of normal subgroup of $S_n \wr S_n$ by $N(S_n \wr S_n)$. Subgroup with number $i$ from $N(S_n \wr S_n)$ is denoted by $N_i(S_n \wr S_n)$. Subgroup with number $i$ from $N(S_n \wr S_n)$ is denoted by $N_i(S_n \wr S_n)$.

Theorem 14. The full list of normal subgroups of $S_n \wr S_n \wr S_n$ consists of 50 normal subgroups. These subgroups are the following:

1) **Type $T_{023}$** contains: $E \wr \tilde{A}_n \wr H$, $\tilde{T}_n \wr H$, where $H \in \{\tilde{A}_n, \tilde{A}_n^2, S_n\}$. There are 6 subgroups.

2) **The second type of subgroups is subclass in $T_{023}$ with new base of wreath product subgroup $\tilde{A}_n^2$**: $E \wr S_n \wr \tilde{A}_n^2$, $E \wr A_n \wr \tilde{A}_n^2$, $E \wr N_i(S_n \wr S_n)$. Therefore this class has 12 new subgroups. Thus, the total number of normal subgroups in **Type $T_{023}$** is 18.

3) **Type $T_{003}$**: $A_{003(n^2)} = E \wr E \wr \tilde{A}_n^2$, $\tilde{T}_n$, $\tilde{T}_n^{(3)}$. Hence, here are 3 new subgroups.

4) **Type $T_{123}$**: $N_i(S_n \wr S_n) \wr S_n$, $N_i(S_n \wr S_n) \wr \tilde{A}_n$ and $N_i(S_n \wr S_n) \wr \tilde{A}_n^2$, $N_i(S_n \wr S_n) \wr \tilde{A}_n^2$. Thus, there are 29 new normal subgroups in $T_{123}$, taking into account repetition 3.

References


