Normal subgroups of iterated wreath products of symmetric groups and alternating with symmetric groups

R. Skuratovskii
(Kyiv, National Aviation University, Ukraine)
E-mail: ruslcomp@gmail.com, ruslan.skuratovskii@nau.edu.ua

In this research we continue our previous investigation of wreath product normal structure [1].

The lattice of normal subgroups and their properties for finite iterated wreath products $S_{n_1} \wr \ldots \wr S_{n_m}$, $n, m \in \mathbb{N}$ are found. Special classes of normal subgroups and their orders and generators are found. Further, the monolith of these wreath products is investigated by us.

Let $k(\pi)$ be the number of cycles in decomposition of permutation π of degree n.

The number $n - k(\pi)$ is denoted by $dec(\pi)$, and is called a decrement [2] of permutation π.

As well known [2] the minimal number of transpositions in factorization of a permutation π on transpositions is happen to be equal to $dec(\pi)$. We set $dec(e) = 0$. Therefore the decrement of n-cycle is $n - 1$.

If $\pi_1, \pi_2 \in S_n$, then the following formula holds:

$$dec(\pi_1 \cdot \pi_2) = dec(\pi_1) + dec(\pi_2) - 2m, m \in \mathbf{N}, \tag{1}$$

where m is number of joint simplifying transpositions in π_1 and π_2.

The trivial subgroup of S_n we denote by E.

Definition 1. The set of elements from $S_n \wr S_n, n \geqslant 5$ or $n = 3$ of the tableaux form: $[e]_1, [a_1, a_2, \ldots, a_n]_2$, satisfying the following condition

$$\sum_{i=1}^{n} dec([a_i]_2) = 2k, k \in \mathbb{N}, \tag{2}$$

we will call set of type $\widetilde{A}^{(2)}$ and denote this set by $E \wr \widetilde{A}_n$. For brevity of notation this subgroup be also denoted by $\widetilde{A}_n^{(2)}$. It follows directly from the definition that the set of these elements supplemented by the operation of multiplication in the subdirect product, coincides with the group $E \rtimes (\underbrace{S_n \boxtimes S_n \boxtimes S_n \ldots \boxtimes S_n}_{n})$, where subdirect product satisfies to condition (2).

We remind that the intersection of all non-trivial normal subgroups $Mon(G)$ of G is called the monolith of a group G.

Proposition 2. *Elements of first type form the subgroup $e \wr A_n$. This subgroup is the* ***monolith of*** *$S_n \wr S_n$.*

Now we can recursively construct easiest and elegant subgroup $E \wr \widetilde{A}_n^{(2)}$ of $S_n \wr S_n \wr S_n$.

Definition 3. The subgroup $E \wr \widetilde{A}_n^{(2)}$ be denoted by $\widetilde{A}_n^{(3)}$.

The order of $E \wr \widetilde{A}_n^{(2)}$ is $(n!)^{3n} : 2^3$. Furthermore we prove that $E \wr \widetilde{A}_n^{(2)} \lhd S_n \wr S_n \wr S_n$.

Let the set of elements from $S_n \wr S_n \wr S_n, n \geqslant 3$ of the form:

$$[e]_1, [e, e, \ldots, e]_2, [a_1, a_2, \ldots, a_{n^2}]_3$$

satisfying the following condition

$$\sum_{i=1}^{n^2} dec([a_i]_3) = 2k, k \in \mathbb{N}, \tag{3}$$

be denoted by $\widetilde{A}_{n^2}^{(3)}$.

Proposition 4. *The set of elements of type $\widetilde{A}_{n^2}^{(3)}$ forms a subgroup in $S_n \wr S_n \wr S_n$. Moreover $\widetilde{A}_{n^2}^{(3)} \lhd S_n \wr S_n \wr S_n$.*

Remark 5. We note that $\widetilde{A}_n^{(3)} < \widetilde{A}_{n^2}^{(3)}$. The order of $\widetilde{A}_{n^2}^{(3)}$ is $(n!)^{n^2} : 2$. Furthermore $\widetilde{A}_n^{(3)} \lhd S_n \wr S_n \wr S_n$.

Definition 6. *A subgroup in $S_n \wr S_n$ is called $\widetilde{T_n}$ if it consists of:*

1) *elements of $E \wr A_n$,*
2) *elements with the tableau* [3] *presentation $[e]_1$, $[\pi_1, \ldots, \pi_n]_2$, that $\pi_i \in S_n \setminus A_n$.*

One easy can validates a correctness of this definition, i.e. that the set of such elements form a subgroup and its normality. This subgroup has structure $\tilde{T_n} \simeq (\underbrace{A_n \times A_n \times \cdots \times A_n}_{n}) \rtimes C_2 \simeq \underbrace{S_n \boxplus S_n \ldots \boxplus S_n}_{n}$, where the operation of a subdirect product $\boxplus$ is subject to items 1) and 2).

Definition 7. A subgroup in $S_n \wr S_n \wr S_n$ is of the type $\widetilde{T}_{n^2}^{(3)}$ if it consists of:

1) elements of the form $E \wr E \wr A_n$,
2) elements with the tableau [3] presentation $[e]_1$, $[e \ldots, e]_2$, $[\pi_1 \ldots, \pi_n, \pi_{n+1} \ldots, \pi_{n^2}]_3$, wherein $\forall i = 1, \ldots, n$: $\pi_i \in S_n \setminus A_n$.

We define recursively the subgroup $\widetilde{T}_n^{(3)}$ having n different intervals of elements with the same parity permutations on X^2.

Definition 8. The subgroup of $S_n \wr S_n \wr S_n$ having structure $E \wr \widetilde{T_n}$ is denoted by $\widetilde{T}_n^{(3)}$. The following isomorphism $\widetilde{T}_n^{(3)} \simeq \underbrace{S_n \boxplus S_n \ldots \boxplus S_n}_{n} \times \underbrace{S_n \boxplus S_n \ldots \boxplus S_n}_{n} \times \ldots \times \underbrace{S_n \boxplus S_n \ldots \boxplus S_n}_{n}$, where a tuple $S_n \boxplus S_n \ldots \boxplus S_n$ repeats n times, holds. The operation of a subdirect product $\boxplus$ is determined by Definition 6.

The operation $\boxplus$ accords with the properties described in item 1 and 2 of Definition 6, also $\boxplus$ is determined by automorphism in $\tilde{T_n} \simeq (\underbrace{A_n \times A_n \times \cdots \times A_n}_{n}) \rtimes C_2$ in this case.

Remark 9. Note that in $\widetilde{T}_n^{(3)}$ vertex permutation of tableau third part satisfy the condition: elements with the tableau presentation $[e]_1$, $[e \ldots, e]_2$, $[\pi_1 \ldots, \pi_n; \pi_{n+1} \ldots, \pi_{n^2}]_3$, that either all $\pi_i \in S_n \setminus A_n$ or all $[\pi_i]_3 \in A_n$ for $1 < i \leq n$, $n + 1 \leq i < 2n$, ... , $n^2 - n < i \leq n^2$.

Here are the names of (almost all) predefined theorem-like environments.

Proposition 10. *The subgroup $E \wr A_n$ is the **monolith of** $S_n \wr S_n$.*

We call level of $AutX^*$ as active if it has at least one non-trivial permutation. Denote by $Aut_f X^*$ the group of all finite automorphism of spherically homogeneous rooted tree.

Proposition 11. *Let $H \lhd Aut_f X^*$ with depth k then H contains k-th level subgroup P having all even vertex permutations $p_{ki} \in A_n$ on X^k and trivial permutations in vertices of rest of levels. Furthermore P is normal in W provided k is last active level of $Aut_f X^*$.*

Theorem 12. *Proper normal subgroups in $S_n \wr S_m$ (action of group is left), where $n, m \geq 3$ with $n, m \neq 4$ are of the following types:*

1) *the subgroups of the first level stabilizer* [1, 4] *are*

$$E \wr \widetilde{A_m},\ \widetilde{T_m},\ E \wr S_m, E \wr A_n,$$

2) *the subgroups that act on both levels are $A_n \wr \widetilde{A_m}$, $S_n \wr \widetilde{A_m}$, $A_n \wr S_m$,*

wherein the subgroup $S_n \wr \widetilde{A_m} \simeq S_n \leftthreetimes (\underbrace{S_m \boxtimes S_m \boxtimes S_m \ldots \boxtimes S_m}_{n})$ endowed with the subdirect product [4] *satisfying to condition* (3)*, moreover $S_n \wr \widetilde{A_m}$ has two isomorphic copies, embedded into $S_n \wr S_m$ in different ways.*

In total there are 8 proper normal subgroups in $S_n \wr S_m$.

Proposition 13. *All normal subgroups of $S_n \wr (S_m \times S_k)$ can be partitioned in 2 types:*

1) $E \wr (N_i \times N_j)$*, where* $N_i \triangleleft \prod\limits_{k=1}^{n} S_m^{(k)}$ *and* $N_j \triangleleft \prod\limits_{l=1}^{n} S_l^{(l)}$.
2) $\widetilde{A}_l \wr (N_i \times N_j)$*, where $\widetilde{A}_i \triangleleft S_n$, N_i and N_j are subgroups from item 1) possessing an extension by $\widetilde{A}_i$ in a correspondent groups $S_n \wr S_m$ and in $S_n \wr S_k$. The full list of them:* $S_n \wr \left(S_m \times \widetilde{A}_k\right)$, $S_n \wr \left(\widetilde{A}_m \times \widetilde{A}_k\right)$, $S_n \wr \left(\widetilde{A}_m \times S_k\right)$, *also* $A_n \wr \left(S_m \times \widetilde{A}_k\right)$, $A_n \wr \left(\widetilde{A}_m \times \widetilde{A}_k\right)$, $A_n \wr \left(\widetilde{A}_m \times S_k\right)$.

We denote the set of normal subgroup of $S_n \wr S_n$ by $N(S_n \wr S_n)$. Subgroup with number i from $N(S_n \wr S_n)$ is denoted by $N_i(S_n \wr S_n)$.

Theorem 14. *The full list of normal subgroups of $S_n \wr S_n \wr S_n$ consists of 50 normal subgroups. These subgroups are the following:*

1) ***Type*** T_{023} *contains:* $E \wr \tilde{A}_n \wr H$, $\widetilde{T_n} \wr H$, *where* $H \in \{\tilde{A}_n, \tilde{A}_{n^2}, S_n\}$. *There are 6 subgroups.*
2) ***The second type of subgroups is subclass in*** T_{023} *with new base of wreath product subgroup* $\tilde{A}_{n^2}$*:* $E \wr S_n \wr \tilde{A}_{n^2}$, $E \wr A_n \wr \tilde{A}_{n^2}$ $E \wr N_i(S_n \wr S_n)$. *Therefore this class has 12 new subgroups. Thus, the total number of normal subgroups in* ***Type*** T_{023} *is 18.*
3) ***Type*** T_{003}*:* $A^{(3)}_{00(n^2)} = E \wr E \wr \tilde{A}_{n^2}, \widetilde{T_{n^2}}, \widetilde{T_n}^{(3)}$. *Hence, here are 3 new subgroups.*
4) ***Type*** T_{123}*:* $N_i(S_n \wr S_n) \wr S_n$, $N_i(S_n \wr S_n) \wr \tilde{A}_n$ *and* $N_i(S_n \wr S_n) \wr \tilde{A}_{n^2}$. *Thus, there are 29 new normal subgroups in T_{123}, taking into account repetition* [5].

References

[1] *Skuratovskii R.V.* Invariant structures of wreath product of symmetric groups. Naukovuy Chasopus of Science hour writing of the National Pedagogical University named after M.P. Dragomanova. (in ukrainian) Series 01. Physics and Mathematics. — 2009. Issue 10. — P. 163 – 178.

[2] *Sachkov, V.N.*, Combinatorial methods in discrete Mathematics. Encyclopedia of mathematics and its applications 55. Cambridge Press. 2008. P. 305.

[3] *Kaloujnine L. A.* Sur les p-group de Sylow du groupe symétrique du degré p^m. // C. R. Acad. Sci. Paris. — 1945. — **221**. — P. 222–224.

[4] *Drozd, Y.A., Skuratovskii R.V.* Generators and relations for wreath products of groups. *Ukr. Math. J.* (2008), *60*, pp. 1168–1171.

[5] *Ruslan Skuratovskii.* Normal subgroups of iterated wreath products of symmetric groups and alternating with symmetric groups. 2022, Source: [https://doi.org/10.48550/arXiv.2108.03752]