ASYMPTOTIC BEHAVIOR OF THE WIDTHS OF CLASSES OF THE GENERALIZED POISSON INTEGRALS

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Let L_p, $1 \le p \le \infty$, and C be the spaces of 2π–periodic functions with standard norms $\|\cdot\|_{L_p}$ and $\|\cdot\|_C$, respectively.

Denote by $C^{\psi}_{\bar{\beta},p}$, $1 \le p \le \infty$, the set of all 2π-periodic functions f, representable as convolution

$$f(x) = \frac{a_0}{2} + \frac{1}{\pi}\int\limits_{-\pi}^{\pi} \varphi(x-t)\Psi_{\bar{\beta}}(t)dt, \quad a_0 \in \mathbb{R}, \quad \varphi \in B^0_p = \{g \in L_p: \ \|g\|_p \le 1, \ g \perp 1\}, \tag{1}$$

with a fixed generated kernel $\Psi_{\bar{\beta}} \in L_{p'}$, $1/p + 1/p' = 1$, the Fourier series of which has the form

$$S[\Psi_{\bar{\beta}}](t) = \sum_{k=1}^{\infty} \psi(k) \cos\left(kt - \frac{\beta_k \pi}{2}\right), \quad \beta_k \in \mathbb{R}, \quad \psi(k) \ge 0.$$

A function f in the representation (1) is called $(\psi, \bar{\beta})$-integral of the function φ and is denoted by $\mathcal{J}^{\psi}_{\bar{\beta}}\varphi$ $(f = \mathcal{J}^{\psi}_{\bar{\beta}}\varphi)$. If $\psi(k) \ne 0$, $k \in \mathbb{N}$, then the function φ in the representation (1) is called $(\psi, \bar{\beta})$-derivative of the function f and is denoted by $f^{\psi}_{\bar{\beta}}$ $(\varphi = f^{\psi}_{\bar{\beta}})$. The concepts of $(\psi, \bar{\beta})$-integral and $(\psi, \bar{\beta})$-derivative was introduced by Stepanets [1]. Since $\varphi \in L_p$ and $\Psi_{\bar{\beta}} \in L_{p'}$, then the function f of the form (1) is a continuous function, i.e. $C^{\psi}_{\bar{\beta},p} \subset C$ (see [1, Proposition 3.9.2.]).

In the case $\beta_k \equiv \beta$, $\beta \in \mathbb{R}$, the classes $C^{\psi}_{\bar{\beta},p}$ are denoted by $C^{\psi}_{\beta,p}$.

For $\psi(k) = k^{-r}, r > 0$, the classes $C^{\psi}_{\bar{\beta},p}$ and $C^{\psi}_{\beta,p}$ are denoted by $W^r_{\bar{\beta},p}$ and $W^r_{\beta,p}$, respectively. The classes $W^r_{\beta,p}$ are the well-known Weyl-Nagy classes (see [1]). In other words, $W^r_{\beta,p}, 1 \le p \le \infty$, are the classes of 2π-periodic functions f, representable as convolutions of the Weyl-Nagy kernels $B_{r,\beta}(t) = \sum\limits_{k=1}^{\infty} k^{-r} \cos\left(kt - \frac{\beta\pi}{2}\right)$, $r > 0$, $\beta \in \mathbb{R}$, with functions $\varphi \in B^0_p$.

If $r \in \mathbb{N}$ and $\beta = r$, then the functions $B_{r,\beta}$ are the well-known Bernoulli kernels and the corresponding classes $W^r_{\beta,p}$ coincide with the well-known classes W^r_p which consist of 2π-periodic functions f with absolutely continuous derivatives $f^{(k)}$ up to $(r-1)$-th order inclusive and such that $\|f^{(r)}\|_p \le 1$.

For $\psi(k) = e^{-\alpha k^r}$, $\alpha > 0$, $r > 0$, the classes $C^{\psi}_{\bar{\beta},p}$ and $C^{\psi}_{\beta,p}$ are denoted by $C^{\alpha,r}_{\bar{\beta},p}$ and $C^{\alpha,r}_{\beta,p}$, respectively. The sets $C^{\alpha,r}_{\beta,p}$ are well-known classes of the generalized Poisson integrals [1], i.e. classes of convolutions with the generalized Poisson kernels

$$P_{\alpha,r,\beta}(t) = \sum_{k=1}^{\infty} e^{-\alpha k^r} \cos\left(kt - \frac{\beta\pi}{2}\right), \quad \alpha > 0, \quad r > 0, \quad \beta \in \mathbb{R}.$$

Further, let K be a convex centrally symmetric subset of C and let B be a unit ball of the space C. Let also F_N be an arbitrary N-dimensional subspace of space C, $N \in \mathbb{N}$, and $\mathscr{L}(C, F_N)$ be a set of linear operators from C to F_N. By $\mathscr{P}(C, F_N)$ denote the subset of projection operators of the set

$\mathscr{L}(C, F_N)$, that is, the set of the operators A of linear projection onto the set F_N such that $Af = f$ when $f \in F_N$. The quantities

$$b_N(K,C) = \sup_{F_{N+1}} \sup\{\varepsilon > 0 : \varepsilon B \cap F_{N+1} \subset K\},$$

$$d_N(K,C) = \inf_{F_N} \sup_{f\in K} \inf_{u\in F_N} \|f-u\|_C,$$

$$\lambda_N(K,C) = \inf_{F_N} \inf_{A\in\mathscr{L}(C,F_N)} \sup_{f\in K} \|f - Af\|_C,$$

$$\pi_N(K,C) = \inf_{F_N} \inf_{A\in\mathscr{P}(C,F_N)} \sup_{f\in K} \|f - Af\|_C,$$

are called Bernstein, Kolmogorov, linear, and projection N-widths of the set K in the space C, respectively.

The results containing order estimates of the widths b_N, d_N, λ_N or π_N in the case of $K = C^{\psi}_{\bar\beta,p}$ (and, in particular, $W^r_{\beta,p}$ and $C^{\psi}_{\beta,p}$) can be found, for example, in the monographs of Tikhomirov, Pinkus, Kornejchuk, Romanyuk, Temlyakov etc.

Theorem 1. *Let* $\bar\beta = \{\beta_k\}_{k=1}^{\infty}, \beta_k \in \mathbb{R}$, $\alpha > 0$, $r > 1$, $n \in \mathbb{N}$ *and be such that*

$$(n-1)^r > \frac{1}{\alpha}, \tag{2}$$

then the following inequalities hold

$$\frac{1}{\sqrt{\pi}} e^{-\alpha n^r} \left(1 - \frac{2\gamma_{\alpha,r,n} e^{-2\alpha r(n-1)^{r-1}}}{1 + 2\gamma_{\alpha,r,n} e^{-2\alpha r(n-1)^{r-1}}}\right)^{\frac{1}{2}} \le P_{2n}(C^{\alpha,r}_{\bar\beta,2}, C)$$

$$\le P_{2n-1}(C^{\alpha,r}_{\bar\beta,2}, C) \le \frac{1}{\sqrt{\pi}} e^{-\alpha n^r} \left(1 + e^{-2\alpha r n^{r-1}} \left(1 + \frac{1}{2\alpha r n^{r-1}}\right)\right)^{\frac{1}{2}}, \tag{3}$$

where P_N *is any of the widths* b_N, d_N, λ_N *or* π_N *and*

$$\gamma_{\alpha,r,n} = \left(1 + \frac{1}{\alpha r(n-1)^{r-1}} + e^{-2\alpha(n-1)^r} \max\left\{e^{4\alpha}, \frac{e^2}{\alpha^{1+1/r}}\right\}\right). \tag{4}$$

Theorem 2. *Let* $\bar\beta = \{\beta_k\}_{k=1}^{\infty}, \beta_k \in \mathbb{R}$, $\alpha > 0$, $r > 1$, $n \in \mathbb{N}$ *and the condition* (2) *is satisfied. Then as* $n \to \infty$ *the following asymptotic equalities hold*

$$\left.\begin{array}{l} P_{2n}(C^{\alpha,r}_{\bar\beta,2}, C) \\ P_{2n-1}(C^{\alpha,r}_{\bar\beta,2}, C) \end{array}\right\} = e^{-\alpha n^r} \left(\frac{1}{\sqrt{\pi}} + \mathcal{O}(1)\gamma_{\alpha,r,n} e^{-\alpha r(n-1)^{r-1}}\right), \tag{5}$$

where P_N *is any of the widths* b_N, d_N, λ_N *or* π_N *and* $\gamma_{\alpha,r,n}$ *is defined by* (4) *and* $\mathcal{O}(1)$ *are the quantities uniformly bounded in all parameters.*

Note that the Theorem 2 complements the results of the works of Shevaldin (1992), Stepanets and Serdyuk (1995), Serdyuk (1999), Serdyuk and Sokolenko (2011), Serdyuk and Bodenchuk (2013), which contain exact estimates for the widths of the classes of convolutions with classical or generalized Poisson kernels.

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References

[1] A.I. Stepanets. *Methods of Approximation Theory.* Utrecht: VSP, 2005.