The Riemann-Hilbert problem and holomorphic bundles framed along a real hypersurface

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The Riemann sphere $\mathbb{P}^1_{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ decomposes as the union $\mathbb{P}^1_{\mathbb{C}} = \bar{U}^- \cup \bar{U}^+$ of two closed disks $\bar{U}^- = \bar{D}$, $\bar{U}^+ = \mathbb{P}^1 \setminus D$ intersecting along their boundary $\partial \bar{U}^{\pm} = S^1$. The Riemann-Hilbert problem, as stated by Hilbert in [2, Kapitel X], asks:

The Riemann-Hilbert Problem. Let $\Gamma : S^1 \to \mathrm{GL}(r, \mathbb{C})$ be a smooth map. Find the pairs (Y^-, Y^+) of continuous maps $Y^{\pm} : \bar{U}^{\pm} \to \mathbb{C}^r$ which are holomorphic on $U^{\pm}$ and satisfy the condition $Y^+|_{S^1} = \Gamma Y^-|_{S^1}$.

More generally, consider

(1) A representation $\rho : G \to \mathrm{GL}(V)$ of a complex Lie group G on a finitely dimensional complex vector space V,
(2) A map $\Gamma : S \to G$ of class $\mathcal{C}^\kappa$ with $\kappa \in [0, \infty]$,
(3) An integer $m \in \mathbb{Z}$ and a V-valued polynomial $\gamma \in V[z]$. Put $d := \deg(\gamma) \in \mathbb{Z}_{\geq -1}$.

Regarding ∞ as an effective divisor on $\mathbb{P}^1_{\mathbb{C}}$, γ can be interpreted as an element of $H^0(\mathcal{O}(d\infty)_{(d+1)\infty} \otimes V)$. We ask:

The general RH problem on $\mathbb{P}^1$. Find the space of pairs (Y^-, Y^+) of continuous maps

$$Y^- : \bar{U}^- \to V,\ Y^+ : \bar{U}^+ \setminus \{\infty\} \to V$$

with Y^- holomorphic on U^-, Y^+ holomorphic on $U^+ \setminus \{\infty\}$ such that $Y^+_S = \rho(\Gamma) Y^-_S$, and

$$\lim_{z \to \infty} (z^{d-m} Y^+(z) - \gamma(z)) = 0$$

The geometric interpretation of the latter condition: Y^+ extends as a section of the sheaf $\mathcal{O}(m\infty) \otimes_{\mathbb{C}} V$ on U^+ whose image in $H^0(\mathcal{O}(m\infty)_{(d+1)\infty} \otimes V)$ via the obvious morphism is $z^{m-d} \otimes \gamma$. Hilbert's original problem is obtained taking ρ to be the canonical representation of $\mathrm{GL}(r, \mathbb{C})$ on $\mathbb{C}^n$, $m = 0$, and $\gamma = 0$.

Complex geometric point of view: Consider the sheaf $\mathcal{V}^\Gamma$ of *local solutions* of the RH problem with $m = \gamma = 0$; this sheaf is given explicitly by:

$$W \mapsto \left\{ \begin{pmatrix} f^- \\ f^+ \end{pmatrix} \in \begin{matrix} \mathcal{C}^0(W \cap \bar{U}^-, V) \\ \times \\ \mathcal{C}^0(W \cap \bar{U}^+, V) \end{matrix} \;\middle|\; \begin{matrix} f^+|_{W \cap S} = \rho(\Gamma)\, f^-|_{W \cap S}, \\ f^{\pm} \text{ is holomorphic on } W \cap U^{\pm} \end{matrix} \right\}.$$

Theorem 1. *Suppose $\kappa \in (1, \infty] \setminus \mathbb{N}$. The sheaf of $\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}$-modules $\mathcal{V}^\Gamma$ is locally free of rank* $\dim(V)$ *and coincides with the apparently smaller sheaf*

$$\left\{ \begin{pmatrix} f^- \\ f^+ \end{pmatrix} \in \begin{matrix} \mathcal{C}^\kappa(W \cap \bar{U}^-, V) \\ \times \\ \mathcal{C}^\kappa(W \cap \bar{U}^+, V) \end{matrix} \;\middle|\; \begin{matrix} f^+|_{W \cap S} = \rho(\Gamma)\, f^-|_{W \cap S}, \\ f^{\pm} \textit{ is holomorphic on } W \cap U^{\pm} \end{matrix} \right\}.$$

We have an obvious identification

$$H^0(\mathcal{O}(d\infty)_{(d+1)\infty} \otimes V) \xrightarrow{\simeq} H^0(\mathcal{V}^\Gamma(d\infty)_{(d+1)\infty}),$$

so γ gives an element $\nu^\Gamma_\gamma \in H^0(\mathcal{V}^\Gamma(d\infty)_{(d+1)\infty})$.

Consider the short exact sequence of coherent sheaves on $\mathbb{P}^1_{\mathbb{C}}$

$$0 \to \mathcal{V}^\Gamma((m-d-1)\infty) \to \mathcal{V}^\Gamma(m\infty) \to \mathcal{V}^\Gamma(m\infty)_{(d+1)\infty} \to 0$$

and the associated cohomology long exact sequence.

Corollary 2. (1) *The space of solutions of the general RH problem is non-empty if and only if the image of* $z^{m-d}\otimes\nu^\Gamma_\gamma$ *via the connecting morphism* $H^0(\mathcal{V}^\Gamma(m\infty)_{(d+1)\infty}) \to H^1(\mathcal{V}^\Gamma((m-d-1)\infty))$ *vanishes.*

(2) *If this is the case, this space has the natural structure of an affine space with model space* $H^0\big(\mathcal{V}^\Gamma((m-d-1)\infty)\big)$*, and can be identified with the pre-image of* $z^{m-d}\otimes\nu^\Gamma_\gamma$ *via the morphism* $H^0\big(\mathcal{V}^\Gamma(m\infty)\big) \to H^0(\mathcal{V}^\Gamma(m\infty)_{(d+1)\infty})$.

(3) *(Regularity) Any solution* (Y^-, Y^+) *of a RH problem with* Γ *of class* $\mathcal{C}^\kappa$ *is also of class* $\mathcal{C}^\kappa$ *up to the boundary.*

Note that, by Grothendieck's classification theorem, $\mathcal{V}^\Gamma$ splits as a direct sum of invertible sheaves, so $\mathcal{V}^\Gamma \simeq \bigoplus_{j=1}^r \mathcal{O}(n_j)$ with $n_j \in \mathbb{Z}$ and $\sum_{j=1}^r n_j = \deg(\mathcal{V}^\Gamma)$. For the canonical representation of $\mathrm{GL}(r,\mathbb{C})$ on $\mathbb{C}^r$ we have $\deg(\mathcal{V}^\Gamma) = -\deg(\det(\Gamma))$.

The RH problem can be naturally generalized in the framework of Riemann surfaces as follows: we replace $\mathbb{P}^1_{\mathbb{C}}$ by an arbitrary closed Riemann surface X, the circle S^1 by an arbitrary (not necessarily connected, not necessarily separating) oriented closed curve $S \subset X$, and Γ by a map $\Gamma \in \mathcal{C}^\kappa(S,G)$. Let $\widehat{X}_S$ be the Riemann surface with boundary obtained by cutting X along S. The unknown of the RH problem associated with these data is a meromorphic map $Y : \widehat{X}_S \setminus \partial\widehat{X}_S \dashrightarrow V$, which extends continuously around $\partial\widehat{X}_S$ and satisfies a compatibility condition associated with Γ. In this general framework one also has a complex geometric interpretation of the space of solutions which generalizes Corollary 2.

All these results are applications of a general gluing theorem for holomorphic bundles. The same theorem can be used to prove an isomorphism theorem between moduli spaces of framed holomorphic bundles [5], [1]:

Let E be a differentiable vector bundle of rank r on a closed complex manifold X, $S \subset X$ a closed, separating real hypersurface, $X = \bar{X}^- \cup \bar{X}^+$ the corresponding decomposition of X as union of compact complex manifolds with boundary, and $E^\pm := E|_{\bar{X}^\pm}$.

Theorem 3. *The moduli space* $\mathcal{M}_S(E)$ *of* S*-framed holomorphic structures on* E *can be identified with the fibre product of the moduli spaces* $\mathcal{M}_{\partial\bar{X}^\pm}(E^\pm)$ *of boundary framed (formally) holomorphic structures on* $E^\pm$ *over the space of Cauchy-Riemann operators on the trivial bundle of rank* r *on* S*.*

Note that $\mathcal{M}_{\partial\bar{X}^\pm}(E^\pm)$ can be further identified with moduli spaces of boundary framed Hermitian-Einstein connections on $E^\pm$ using a version of the classical Kobayashi-Hitchin correspondence [3] for complex manifolds with boundary [1], [6].

References

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