The Riemann-Hilbert problem and holomorphic bundles framed along a real hypersurface

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The Riemann sphere \( \mathbb{P}^1 = \mathbb{C} \cup \{ \infty \} \) decomposes as the union \( \mathbb{P}^1 = \tilde{U}^- \cup \tilde{U}^+ \) of two closed disks \( \tilde{U}^- = D, \tilde{U}^+ = \mathbb{P}^1 \setminus D \) intersecting along their boundary \( \partial \tilde{U}^\pm = S^1 \). The Riemann-Hilbert problem, as stated by Hilbert in [2, Kapitel X], asks:

**The Riemann-Hilbert Problem.** Let \( \Gamma : S^1 \to \text{GL}(r; \mathbb{C}) \) be a smooth map. Find the pairs \((Y^-, Y^+)\) of continuous maps \(Y^\pm : \tilde{U}^\pm \to \mathbb{C}^r\) which are holomorphic on \(U^\pm\) and satisfy the condition \(Y^+|_{S^1} = \Gamma Y^-|_{S^1}\).

More generally, consider

1. A representation \( \rho : G \to \text{GL}(V) \) of a complex Lie group \( G \) on a finitely dimensional complex vector space \( V \),
2. A map \( \Gamma : S \to G \) of class \( C^\kappa \) with \( \kappa \in [0, \infty) \),
3. An integer \( m \in \mathbb{Z} \) and a \( V \)-valued polynomial \( \gamma \in V[z] \). Put \( d := \deg(\gamma) \in \mathbb{Z}_{\geq 1} \).

Regarding \( \infty \) as an effective divisor on \( \mathbb{P}^1_\mathbb{C} \), \( \gamma \) can be interpreted as an element of \( H^0(O(d\infty)(d+1)\infty \otimes V) \). We ask:

**The general RH problem on \( \mathbb{P}^1 \).** Find the space of pairs \((Y^-, Y^+)\) of continuous maps

\[
Y^{-} : \tilde{U}^- \to V, \ Y^{+} : \tilde{U}^+ \setminus \{\infty\} \to V
\]

with \( Y^- \) holomorphic on \( U^- \), \( Y^+ \) holomorphic on \( U^+ \setminus \{\infty\} \) such that \( Y^+_S = \rho(\Gamma) Y^-_S \), and

\[
\lim_{z \to \infty} (z^{-m} Y^+(z) - \gamma(z)) = 0
\]

**The geometric interpretation of the latter condition:** \( Y^+ \) extends as a section of the sheaf \( O(m\infty) \otimes \mathbb{C} \) on \( U^+ \) whose image in \( H^0(O(m\infty)(d+1)\infty \otimes V) \) via the obvious morphism is \( z^{m-d} \otimes \gamma \). Hilbert’s original problem is obtained taking \( \rho \) to be the canonical representation of \( \text{GL}(r; \mathbb{C}) \) on \( \mathbb{C}^r \), \( m = 0 \), and \( \gamma = 0 \).

**Complex geometric point of view:** Consider the sheaf \( \mathcal{V}^\Gamma \) of local solutions of the RH problem with \( m = \gamma = 0 \); this sheaf is given explicitly by:

\[
W \mapsto \begin{cases} 
\left( f^- \atop f^+ \right) \in C^0(W \cap \tilde{U}^-, V) \times C^0(W \cap \tilde{U}^+, V) & f^+|_{W \cap S} = \rho(\Gamma) f^-|_{W \cap S}, \ f^+ \text{ is holomorphic on } W \cap U^+ \\
\right. 
\end{cases}
\]

**Theorem 1.** Suppose \( \kappa \in (1, \infty) \setminus \mathbb{N} \). The sheaf of \( O_{\mathbb{P}^1_\mathbb{C}} \)-modules \( \mathcal{V}^\Gamma \) is locally free of rank \( \dim(V) \) and coincides with the apparently smaller sheaf

\[
\left\{ \left( f^- \atop f^+ \right) \in C^\kappa(W \cap \tilde{U}^-, V) \times C^\kappa(W \cap \tilde{U}^+, V) & f^+|_{W \cap S} = \rho(\Gamma) f^-|_{W \cap S}, \ f^+ \text{ is holomorphic on } W \cap U^+ \right. \}
\]

We have an obvious identification

\[
H^0(O(d\infty)(d+1)\infty \otimes V) \cong H^0(\mathcal{V}^\Gamma(d\infty)(d+1)\infty),
\]

so \( \gamma \) gives an element \( \nu^\Gamma_\gamma \in H^0(\mathcal{V}^\Gamma(d\infty)(d+1)\infty) \).
Consider the short exact sequence of coherent sheaves on $\mathbb{P}^1_C$
\[
0 \to \mathcal{V}^T((m - d - 1)\infty) \to \mathcal{V}^F(m\infty) \to \mathcal{V}^F(m\infty)_{(d+1)\infty} \to 0
\]
and the associated cohomology long exact sequence.

**Corollary 2.** (1) The space of solutions of the general RH problem is non-empty if and only if the image of $z^{m-d}\otimes\nu^T_\gamma$ via the connecting morphism $H^0(\mathcal{V}^F(m\infty)_{(d+1)\infty}) \to H^1(\mathcal{V}^T((m-d-1)\infty))$ vanishes.
(2) If this is the case, this space has the natural structure of an affine space with model space $\text{Regularity}$ Any solution replaces $\mathcal{V}^F$ continuously around $X_1$.
Let $E$ be a differentiable vector bundle of rank $r$ on a closed complex manifold $X$, $S \subset X$ a closed, separating real hypersurface, $X = X^- \cup X^+$ the corresponding decomposition of $X$ as union of compact complex manifolds with boundary, and $E^\pm := E|_{X^\pm}$.

**Theorem 3.** The moduli space $\mathcal{M}_S(E)$ of $S$-framed holomorphic structures on $E$ can be identified with the fibre product of the moduli spaces $\mathcal{M}_{\partial\bar{X}^\pm}(E^\pm)$ of boundary framed (formally) holomorphic structures on $E^\pm$ over the space of Cauchy-Riemann operators on the trivial bundle of rank $r$ on $S$.

Note that $\mathcal{M}_{\partial\bar{X}^\pm}(E^\pm)$ can be further identified with moduli spaces of boundary framed Hermitian-Einstein connections on $E^\pm$ using a version of the classical Kobayashi-Hitchin correspondence for complex manifolds with boundary.

**References**