About some Steiner trees

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I will talk about the famous Steiner problem.
We denote by $\mathcal{H}^1$ the linear Haurdorff measure (roughly speaking, length).

Problem 1 (Euclidean Steiner problem)**.** Let C be a compact subset of $\mathbb{R}^d$. To find a closed S such that $S \cup C$ is connected and $\mathcal{H}^1(S)$ is minimal.

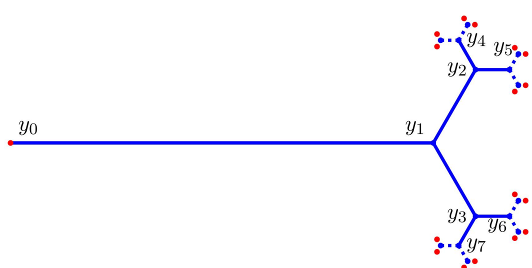
Some properties (if $\mathcal{H}^1(S) < \infty$) hold:

- S exists;
- S contains no loops;
- only two variants of neighbourhoods for points from $S \setminus C$;
- only two variants of a neighbourhood of a point $x \in S \setminus C$:
  - a regular tripod (x is a **branching point**);
  - a segment; x is an inner point.
- S contains at most countable number of branching points.

Usually S is usually called **Steiner tree**, and it is called **indecomposable** (irreducible, full), when $S \setminus C$ is connected. If C is totally disconnected then S should be connected.

Theorem 2 (Paolini–Stepanov–T, 2015; Cherkashin–T. 2023; Paolini–Stepanov 2023)**.** *There is a compact planar set C for which the unique solution of the Steiner problem is indecomposable and has infinite number of triple points.*

In the theorem C and Σ are self-similar fractals with a sufficiently small scale factor.



Theorem 3 (Basok, Cherkashin, T., 2022)**.** *In the plane for $m \geq 4$ the set of m terminals (considered as a subset of $\mathbb{R}^{2m}$) with non unique Steiner trees has the Hausdorff dimension $2m - 1$.*

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