Proximal connectedness. Spatially and descriptively connected spaces

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In this paper, we give results for spatially-connected spaces (X, δ) [5] (a widely-considered proximity space) and descriptively-connected spaces (X, δ_Φ), a recent form of proximity space [3] with a number of applications. Definition 1 is analogous to connectedness in digital topology [2].

Definition 1. Let (X, δ) be a proximity space. Then two nonempty subsets A and B are δ-connected, provided there exists a finite family of subsets $\{E_i\}_{i=0}^n$ of X such that $A = E_0$, $B = E_n$, and $E_i \; \delta \; E_{i+1}$ for all $i = 0, 1, \ldots, n-1$. A proximity space (X, δ) is said to be connected, provided any pair of subsets A, B of X are δ-connected.
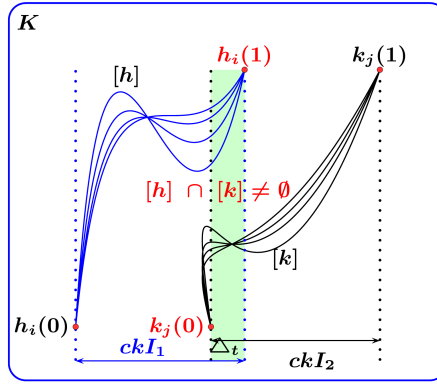

FIGURE 1. Spatially near homotopy maps in spatial far homotopy classes $[h], [k]$

Example 2. In Figure 1, let $h, k : I \to K$ be continuous maps called homotopies in a space K. A homotopic class for different maps h (denoted by $[h] = \{h_0, \ldots, h_i, \ldots, h_{n-1[n]}\}$ with $[n] = \mod n \in \mathbb{Z}^+$) is a collection of $h_{i[n]}$ homotopic maps that have the same endpoints, namely, $h_i(0)$ and $h_i(1)$. The maps in $[h]$ are spatially near. Similarly, the maps in $[k]$ are spatially near. However, the homotopy classes $[h], [k]$ are spatially far. Also, from Definition 1, every pair of nonempty subsets $A, B \in [h]$ are δ-connected, since $A \; \delta \; B$. Likewise, the maps in $[k]$ are δ-connected.

Theorem 3. *Let (X, δ_1) and (Y, δ_2) be proximity spaces. Then a map $f : X \to Y$ is proximally continuous if and only if a pair of δ_1-connected subsets of X is mapped to a pair of δ_2-connected subsets of Y.*

Definition 4. [1] Let (X, δ_Φ) be a descriptive proximity space. Then two subsets A and B are δ_Φ-**connected**, provided there exists a finite family of subsets $\{D_i\}_{i=0}^n$ of X such that $D_0 = A$,

$D_n = B$, and $D_i \, \delta_\Phi \, D_{i+1}$ for all $i = 0, 1, \ldots, n-1$. A descriptive proximity space (X, δ_Φ) is said to be **descriptively connected**, provided any pair of subsets A, B of X are δ_Φ-connected.

Proposition 5. *In a digital topology space X, descriptively near sets in a digital image X are descriptively connected.*

Theorem 6. *Let (X, δ_{Φ_1}) and (Y, δ_{Φ_2}) be descriptive proximity spaces. Then a map $f : X \to Y$ is descriptive proximally continuous if and only a pair of δ_{Φ_1}-connected subsets of X is mapped to a pair of δ_{Φ_2}-connected subsets of Y.*

Given a set $X \subset \mathbb{Q} \times \mathbb{Q}$, a digital image on X is a map $Img : X \to \mathbb{R}$ so that each picture element $p \in X$ aka (sub)pixels or (sub)voxels has location $\mathbb{Q} \times \mathbb{Q}$ and value $Img(p) \in \mathbb{R}$. There are two distinct type digital images, namely, frames (denoted by Img_t or simply by fr_t) which is a time-ordered sequence of images in which each frame occurs at an elapsed time t in a video, and single images (denoted by Img). The intersection of the closures of bounded regions with nonempty interiors give rise to $\underset{\kappa}{\delta}$-connectedness.

Definition 7. Given a digital image Img, two subimages A and B, are adjacent, denoted by $A \underset{\kappa}{\delta} B$, provided there exist pixels $p \in A$ and $q \in B$ such that $p = q$ or p and q are adjacent.

Definition 8. A bounded region $E \subset fr$ with a non-empty interior is said to be $\underset{\kappa}{\delta}$-connected, provided for each pair of distinct voxels p and q in E, there exists a finite sequence of voxels $p = v_0, v_1, \ldots, v_n = q$ such that each pair of consecutive voxels v_i and v_{i+1} is $\underset{\kappa}{\delta}$-connected for all $i = 0, 1, \ldots, n-1$.

Proposition 9. $\underset{\kappa}{\delta}$*-connected regions are preserved under a continuous digital functions.*

Example 10. Consider a moving object appears as a bounded foreground region recorded in a video frame fr_t. Observe that the moving object -as a bounded region- is partitioned into (bounded) $\underset{\kappa}{\delta}$ connected subregions, which cover the moving object with a minimum number of contractible subregions is a form of geometric Lusternik-Schirel'mann category [4].

Spatially near subsets in a digital image (*i.e.*, subsets that share points) reside in a discrete proximity space.

Definition 11. For any pair of nonempty subsets A, B in a digital image Img, A is near B (denoted by $A \, \delta \, B$), provided A and B have points in common, *i.e.*, $A \, \delta \, B$ iff $A \cap B \neq \emptyset$. Hence, δ is a discrete proximity relation and (Img, δ) is a discrete proximity space [6, §40.2, pp. 266-267].

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