Thinness at infinity and Deny's principle of positivity of mass in the theory of Riesz potentials

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This talk is based on [9], and it deals with the theory of potentials on $\mathbb{R}^n$, $n \geqslant 2$, with respect to the Riesz kernel $|x-y|^{\alpha-n}$, $\alpha \in (0,2]$, $\alpha < n$, where $|x-y|$ is the Euclidean distance between $x, y \in \mathbb{R}^n$. Denote by $\mathfrak{M}^+$ the cone of all positive Radon measures μ on $\mathbb{R}^n$ such that the Riesz *potential*

$$U^\mu(x) := \int |x-y|^{\alpha-n}\, d\mu(y)$$

is not identically infinite on $\mathbb{R}^n$, which according to [5, Section I.3.7] occurs if and only if

$$\int_{|y|>1} \frac{d\mu(y)}{|y|^{n-\alpha}} < \infty.$$

Then U^μ is actually finite everywhere on $\mathbb{R}^n$, up to a set of *zero Riesz capacity*, cf. [5, Section III.1.1].

The principle of positivity of mass was first introduced by J. Deny (see e.g. [2]), and for Riesz potentials it reads as follows [3, Theorem 3.11].

Theorem 1. *For any $\mu, \nu \in \mathfrak{M}^+$ such that*

$$U^\mu \leqslant U^\nu \quad \text{everywhere on } \mathbb{R}^n, \tag{1}$$

we have $\mu(\mathbb{R}^n) \leqslant \nu(\mathbb{R}^n)$.

It is easy to verify that (1) can be slightly weakened by replacing 'everywhere on $\mathbb{R}^n$' by 'nearly everywhere on $\mathbb{R}^n$' (see [8, Theorem 2.6], establishing the principle of positivity of mass for potentials with respect to rather general function kernels on locally compact spaces). Recall that a proposition $\mathcal{P}(x)$ is said to hold *nearly everywhere* (*n.e.*) on $A \subset \mathbb{R}^n$ if $c_*(E) = 0$, where E is the set of all $x \in A$ for which $\mathcal{P}(x)$ fails, while $c_*(E)$ denotes the *inner Riesz capacity* of E, see [5, Section II.2.6].

The main result of this talk, given by Theorem 2, shows that Theorem 1 still holds even if (1) is fulfilled only on a proper subset A of $\mathbb{R}^n$, which however must be 'large enough' in an arbitrarily small neighborhood of $\infty_{\mathbb{R}^n}$, the Alexandroff point of $\mathbb{R}^n$. This discovery illustrates a special role of the point at infinity in Riesz potential theory, in particular with regard to the principle of positivity of mass.

Theorem 2. *Given $\mu, \nu \in \mathfrak{M}^+$, assume there exists $A \subset \mathbb{R}^n$ which is not inner α-thin at infinity, and such that*

$$U^\mu \leqslant U^\nu \quad \text{n.e. on } A.$$

Then

$$\mu(\mathbb{R}^n) \leqslant \nu(\mathbb{R}^n).$$

Recall that according to [4, 7], $A \subset \mathbb{R}^n$ is said to be *inner α-thin at infinity* if

$$\sum_{k \in \mathbb{N}} \frac{c_*(A_k)}{q^{k(n-\alpha)}} < \infty, \tag{2}$$

where $q \in (1, \infty)$ and $A_k := A \cap \{x \in \mathbb{R}^n : \ q^k \leqslant |x| < q^{k+1}\}$; or equivalently, if either A is bounded, or $x = 0$ is an inner α-irregular boundary point for the inverse of A with respect to $|x| = 1$. (For the concept of inner α-irregular points for arbitrary $A \subset \mathbb{R}^n$ and relevant results, see [6, Section 6];

compare with [5, Section V.1], where A was required to be Borel.) We emphasize that *if A is not inner α-thin at infinity, then necessarily $c_*(A) = \infty$; but not the other way around* (see [7, Section 2]).

The following theorem shows that Theorem 2 is *sharp* in the sense that the requirement on A of not being α-thin at infinity can not in general be weakened.

Theorem 3. *If $A \subset \mathbb{R}^n$ is inner α-thin at infinity, then there are $\mu_0, \nu_0 \in \mathfrak{M}^+$ such that $U^{\mu_0} = U^{\nu_0}$ nearly everywhere on A, but nonetheless, $\mu_0(\mathbb{R}^n) > \nu_0(\mathbb{R}^n)$.*

Nevertheless, Theorem 2 remains valid for *arbitrary* $A \subset \mathbb{R}^n$ once we impose upon $\mu, \nu \in \mathfrak{M}^+$ suitable additional requirements (see Theorem 4 below).

A measure $\mu \in \mathfrak{M}^+$ is said to be *carried* by $A \subset \mathbb{R}^n$ if $\mathbb{R}^n \setminus A$ is μ-negligible, or equivalently if A is μ-measurable and $\mu = \mu|_A$, $\mu|_A$ being the trace of μ to A, cf. [1, Section V.5.7]. We denote by $\mathfrak{M}^+_A$ the cone of all $\mu \in \mathfrak{M}^+$ carried by A. (For closed A, μ is carried by A if and only if it is supported by A.)

A measure $\mu \in \mathfrak{M}^+$ is said to be *C-absolutely continuous* if $\mu(K) = 0$ for every compact set $K \subset \mathbb{R}^n$ of zero Riesz capacity. This certainly occurs if $\int U^\mu \, d\mu$ is finite (or, more generally, if U^μ is locally bounded); but not conversely, see [5, pp. 134–135].

Theorem 4. *For any set $A \subset \mathbb{R}^n$ and any C-absolutely continuous measures $\mu, \nu \in \mathfrak{M}^+_A$ such that $U^\mu \leqslant U^\nu$ n.e. on A, we still have $\mu(\mathbb{R}^n) \leqslant \nu(\mathbb{R}^n)$.*

Remark 5. If $A \cap A_I = \varnothing$, where A_I denotes the set of all inner α-irregular points for A, then the requirement of C-absolute continuity imposed on μ and ν, is unnecessary for the validity of Theorem 4.

Remark 6. The proofs of the above-quoted theorems are based on the theory of inner α-Riesz balayage as well as on that of inner α-Riesz equilibrium measures, both originated in [6, 7] (see also [8]). The concept of inner equilibrium measure is understood in an extended sense where its energy as well as its total mass may be infinite. The following two facts of these theories are crucial to our proofs:

- $A \subset \mathbb{R}^n$ is not α-thin at infinity if and only if the inner balayage of *any* $\mu \in \mathfrak{M}^+$ to A preserves its total mass (see [7, Corollary 5.3]).
- The inner α-Riesz equilibrium measure of $A \subset \mathbb{R}^n$ exists if and only if A is α-thin at infinity (see [7, Theorem 2.1]).

Remark 7. The results presented in the talk have already found applications to minimum Riesz energy problems in the presence of external fields, see for instance [10, Section 4.10].

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