On the category of representations of a third-order semigroup without unit and zero elements

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The classification of the semigroups of third order (in terms of Cayley tables, up to isomorphism and antiisomorphism, was first received by T. Tamura in 1953 [1]. The minimal systems of generators and the corresponding defining relations for all such semigroups were described by V. M. Bondarenko and Y. V. Zatsikha in [2].

If one considers only commutative semigroups and only those that are not neither cyclic nor cyclic with an attached unit or zero element, then there exist, up to isomorphism, the following four semigroups (in square brackets are indicated all elements, in angular a minimal system of generators, and then the defining relations; the trivial relations for unit and zero generating elements 0 and e are not written out):

(a) $(0, b, c) = \langle b, c\rangle$: $b^2 = 0$, $c^2 = 0$, $bc = cb = 0$;

(b) $(0, b, c) = \langle b, c\rangle$: $b^2 = b$, $c^2 = c$, $bc = cb = 0$;

(c) $(0, b, c) = \langle b, c\rangle$: $b^2 = 0$, $c^2 = c$, $bc = cb = 0$;

(d) $(c^2, b, c) = \langle b, c\rangle$: $b^3 = b^2$, $c^3 = c$
(is a consequence of the remaining relations),
$b^2 = c^2$, $bc = cb = c$.

Except for the semigroup (a), all these semigroups are of finite representation type over any field K [2], and in this case one of the forms of studying the category of representations is the description of the Auslander algebra as the algebra of endomorphisms of the direct sum of representatives of all equivalence classes of indecomposable representations. In the simple case (b), the Auslander algebra was considered as an example in [3], and in the case (c) it was described in [4]. Here we consider the Auslander algebra of the semigroup (d) which is denoted by S_d.

Theorem 1. *The Auslander algebra $Aus_K(S_d)$ of the semigroup S_d over a field K of characteristic not equal to 2 is isomorphic to the algebra of all matrices of the form*

$$X = \begin{pmatrix} x_{11} & 0 & 0 & 0 & 0 \\ 0 & x_{22} & 0 & 0 & 0 \\ 0 & 0 & x_{33} & x_{34} & x_{35} \\ 0 & 0 & 0 & x_{33} & 0 \\ 0 & 0 & 0 & x_{54} & x_{55} \end{pmatrix},$$

where x_{ij} are elements of K.

Theorem 2. *The Auslander algebra $Aus_K(S_d)$ of the semigroup S_d over a field K of characteristic 2 is isomorphic to the algebra of all matrices of the form*

$$X = \begin{pmatrix} x_{11} & x_{12} & x_{13} & 0 & 0 & 0 \\ 0 & x_{11} & 0 & 0 & 0 & 0 \\ 0 & x_{32} & x_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & x_{44} & x_{45} & x_{46} \\ 0 & 0 & 0 & 0 & x_{44} & 0 \\ 0 & 0 & 0 & 0 & x_{65} & x_{66} \end{pmatrix},$$

where x_{ij} are elements of K.

These results were obtained together with Prof. V. M. Bondarenko. The first theorem was published in [5] and the second will be published in [6].

References

[1] T. Tamura. Some remarks on semi-groups and all types of semi-groups of order 2. *J. Gakugei Tokushima Univ.*, 3: 1–11, 1953.

[2] V. Bondarenko, Ja. Zaciha. Canonical forms of matrix representations of small-order semigroups. *Scientific Bulletin of Uzhhorod University, ser. of mathematics and informatics*, 32(1): 36–49, 2018.

[3] V. Bondarenko, O. Zubaruk. Σ-function of the number of parameters for the matrix representations system. *Proc. Inst. math. NAS of Ukraine* 12(3): 56–64, 2015.

[4] O. Zubaruk. On the Auslander algebra of the semigroup generated by two annihilating 2-nilpotent and 2-potent elements. *Scientific Bulletin of Uzhhorod University, ser. of mathematics and informatics*, 38(1): 48–54, 2021.

[5] V. Bondarenko, O. Zubaruk. On the category of representations of the commutative noncyclic semigroup of third order without unic and zero elements. *Scientific Bulletin of Uzhhorod University, ser. of mathematics and informatics*, 41(2): 23–28, 2022.

[6] V. Bondarenko, O. Zubaruk. On the Auslander algebra over a field of characteristic two of the commutative noncyclic semigroup of third order without unit and zero elements. *Scientific Bulletin of Uzhhorod University, ser. of mathematics and informatics*, 42(1), 2023.