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- Differential geometry in the large
- Geometry and topology of differentiable manifolds
- General and algebraic topology
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Hopf-Rinow theorem of sub-Finslerian geometry

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The sub-Finslerian geometry means that the metric F is defined only on a given subbundle of the tangent bundle, called a horizontal bundle. In the paper, a version of the Hopf-Rinow theorem is proved in the case of sub-Finslerian manifolds, which relates the properties of completeness, geodesically completeness, and compactness. The sub-Finsler bundle, the exponential map and the Legendre transformation are deeply involved in this investigation.

We construct a sub-Finsler bundle, which plays a major role in the formalization of the sub-Hamiltonian in sub-Finsler geometry. Moreover, the sub-Finsler bundle allows an orthonormal frame for the sub-Finsler structure. We introduce the notion of an exponential map in sub-Finsler geometry. At the end, our main theorem is stated and proved.

Theorem 1. *Let $(M, \mathcal{D}, F)$ be any connected sub-Finsler manifold, where $\mathcal{D}$ is bracket generating distribution. The following conditions are equivalent:*

(i) *The metric space (M, d) is forward complete.*
(ii) *The sub-Finsler manifold $(M, \mathcal{D}, F)$ is forward geodesically complete.*
(iii) $\Omega_x^* = \mathcal{D}_x^*$, *additionally, the exponential map is onto if there are no strictly abnormal minimizers.*
(iv) *Every closed and forward bounded subset of (M, d) is compact.*

Furthermore, for any $x, y \in M$ there exists a minimizing geodesic γ joining x to y, i.e. the length of this geodesic is equal to the distance between these points.

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Geometric properties of interception curves

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In this study, a plane curve, which was named as *Interception Curve*, was discussed. This curve can be defined in the following way. Suppose one point moves with constant velocity along a straight line, and another point, at the beginning one unit apart from the line and the first point on this line, moves with the same constant speed so that it always stays on a line passing through the first point and the initial position of the second point. This plane curve appears in problems related to the interception of high-speed targets by beam rider missiles (hence the name *Interception Curve*) [2, 5]. This curve was also mentioned in [4, 6, 1]. In [3], at Sect. 1.460 and Sect. 1.507), some methods based on polar and Cartesian coordinates were proposed to find an explicit representation for this curve.

Problem 1. If two points $P(x, y)$ and Q, initially at $O(0, 0)$ and $A(1, 0)$, respectively, move uniformly so that Q is on the line $x = 1$, and P is on the ray OQ then what curve does the point P draw?

Answer. Let us use polar coordinates $r = |OP|$ and $\angle AOQ = \theta$. We obtain ordniary differential equation

$$r(\theta)^2 + (r'(\theta))^2 = \frac{1}{\cos^4\theta}, \tag{1}$$

with initial condition $r(0) = 0$. Note that in the cartesian coordinates, (1) can be written as

$$x^2\sqrt{1 + (y'(x))^2} = y'x - y, \tag{2}$$

with initial condition $y(0) = 0$. By solving this equation, we obtain the parametrization (cf. [3], Sect. 1.507, where the roles of x and y are interchanged)

$$\begin{cases} x(p) = \frac{1}{\sqrt{p}}\int_1^p \frac{\sqrt{t}dt}{2\sqrt{t^2-1}}, \\ y(p) = \frac{\sqrt{p^2-1}}{\sqrt{p}}\int_1^p \frac{\sqrt{t}dt}{2\sqrt{t^2-1}} - \left(\int_1^p \frac{\sqrt{t}dt}{2\sqrt{t^2-1}}\right)^2 \ (p \geq 1). \end{cases} \tag{3}$$

Using all these, the following results are obtained:

Theorem 2. *Suppose that U is the y intercept of the tangent line of the curve (3) at the point P, and this tangent line intersects the line $x = 1$ at point and T. Then*

(1) $x \cdot |UP| = |OU|$,
(2) $\sin\angle QPT = \frac{x^2}{|OP|} = \frac{x}{|OQ|}$,

where x is the abscissa of the point $P(x, y)$.

Theorem 3. *Consider intersection point M of mid-perpendicular of OP and the line perpendicular to UT at the point P. Similarly, conside intersection point N of mid-perpendicular of OQ and the line perpendicular to QT at the point Q. Then the points M and N are equidistant from the point O i.e. $MO{=}NO$.*

The following result shows that there is a connection between the interception curve and Gauss's constant G defined by the arithmetic–geometric mean.

Theorem 4.

$$\lim_{x\to 1^-} |PQ| = \frac{1}{4G^2}.$$

Problem 5. Suppose that two points P and Q, at the beginning at $B(0, 0, 1)$ and $A(1, 0, 0)$, respectively, move uniformly so that Q is on the equator $z = 0$, $x^2 + y^2 = 1$ of sphere $x^2 + y^2 + z^2 = 1$ with center $O(0, 0, 0)$, and P is on the meridian through B and Q of the sphere. What curve does the point P draw?

Answer. We can use spherical coordinates to describe this curve: $\angle AOQ = \theta$ and $\angle POB = \phi$. Since $\rho = |OP| = 1$, for the coordinates of point $P(x, y, z)$, we can write $x = \cos\theta \sin\phi$, $y = \sin\theta \sin\phi$, and $z = \cos\phi$, where we think of $\phi = \phi(\theta)$ as a function of θ. For this curve we obtain

$$\phi = \tan^{-1} \sinh\theta. \tag{4}$$

Note that (4), which can also be expressed as $\sin\phi = \tanh\theta$, is sometimes called Gudermannian function $\mathrm{gd}(x)$. For the curve defined by (4) the following results are obtained.

Theorem 6. $\lim_{\theta\to\infty} |PQ| = 0$.

In the following, we will use notation $\widehat{XY}$ for the spherical distance between points X and Y on a sphere.Of course, for a unit sphere with center O, $\widehat{XY} = \angle XOY$.

Theorem 7. *If a great circle is tangent to the curve (4) at point P, intersects the equator at point T, then*

(1) $\widehat{PT} = \frac{\pi}{2} - \widehat{TQ}$,
(2) $\widehat{TQ} < \widehat{PT}$, *and* $\lim_{\theta\to\infty} \widehat{TQ} = \lim_{\theta\to\infty} \widehat{PT} = \frac{\pi}{4}$.
(3) $\angle BPT = \pi - \widehat{BP}$.

Theorem 8. *If a small spherical circle through point B is tangent to the curve (4) at point P, then its spherical radius R satisfies* $\tan R = \frac{1}{2} \sec^2 \frac{1}{2}\widehat{BP}$.

We can prove some of these results also using simpler plane and spherical geometry methods, which are interesting on their own. It can be shown that the results agree with the angle-preserving property of Mercator and Stereographic projections. The Mercator and Stereographic projections also reveal the symmetry of this curve with respect to Spherical and Logarithmic Spirals.

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Planar and non-planar degenerations with related fundamental groups

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We study planar and non-planar degenerations that are related to algebraic surfaces. It is interesting to see the differences in results and research methods between both cases. We have studied already planar degenerations with an R_k singularity, non-planar degenerations of degree 4, 6, and 8. The fundamental groups of the Galois covers of the related surfaces were investigated, because those groups are invariants of classification of algebraic surfaces in the moduli space.

Theorem 1. *The fundamental groups of surfaces that degenerate to one R_k singularity are all trivial, for any k.*

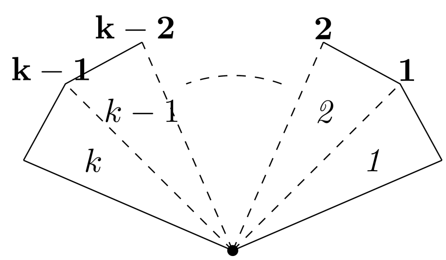

FIGURE 1.1. R_k singularity

Theorem 2. *The fundamental groups of Galois covers related to non-planar degenerations are trivial (for a degree 4 degeneration), $\mathbb{Z}_2^4$ (for a degree 6 degeneration), and a metabelian group of order 2^{23} (for a degree 8 degeneration).*

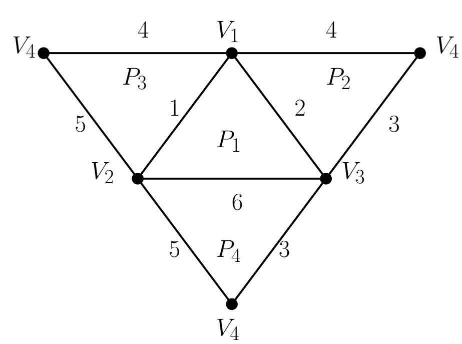

FIGURE 2.2. Degree 4 non-planar degeneration

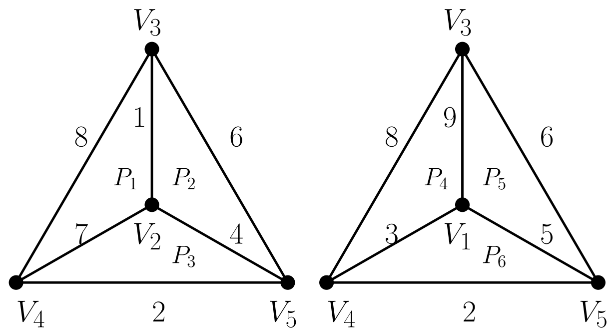

FIGURE 2.3. Degree 6 non-planar degeneration

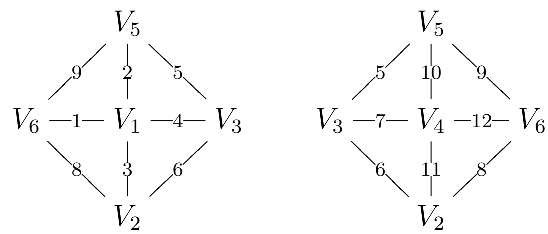

FIGURE 2.4. Degree 8 non-planar degeneration

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Surfaces with zero mean curvature vector in 4-dimensional spaces

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Let (N, h) be an oriented Riemannian 4-manifold. Let $\bigwedge^2 TN$ be the 2-fold exterior power of the tangent bundle TN of N. Then $\bigwedge^2 TN$ is a vector bundle of rank 6 over N and Hodge's $*$-operator gives a bundle decomposition $\bigwedge^2 TN = \bigwedge^2_+ TN \oplus \bigwedge^2_- TN$ by two subbundles $\bigwedge^2_\pm TN$ of rank 3. The twistor spaces associated with N are the sphere bundles in $\bigwedge^2_\pm TN$ and denoted by $U(\bigwedge^2_\pm TN)$. We can refer to [5] for twistor spaces. Let M be a Riemann surface and $F : M \longrightarrow N$ a conformal and minimal immersion. Let F^*TN be the pull-back bundle on M by F. Then F gives its twistor lifts, which are sections of $U(\bigwedge^2_\pm F^*TN)$. Let

σ be the second fundamental form of F. Let w be a local complex coordinate of M and set $\sigma_{ww} := \sigma(\partial/\partial w, \partial/\partial w)$. Then $h(\sigma_{ww}, \sigma_{ww})dw^4$ does not depend on the choice of w and therefore F gives a complex quartic differential Q on M. If N is a 4-dimensional space form, then Q is holomorphic. Isotropicity of F is given by $Q \equiv 0$ and this condition is equivalent to horizontality of a twistor lift of F ([6], [4]).

Let (N, h) be an oriented neutral 4-manifold. Then the metric $\hat{h}$ of $\bigwedge^2 TN$ induced by h has signature $(2, 4)$. We have a bundle decomposition $\bigwedge^2 TN = \bigwedge^2_+ TN \oplus \bigwedge^2_- TN$, and the restriction of $\hat{h}$ on each of $\bigwedge^2_\pm TN$ has signature $(1, 2)$. The space-like (or hyperbolic) twistor spaces associated with N are fiber bundles in $\bigwedge^2_\pm TN$ such that fibers are hyperboloids of two sheets, and denoted by $U_+\big(\bigwedge^2_\pm TN\big)$. We can refer to [3] for space-like twistor spaces. Let M be a Riemann surface and $F : M \longrightarrow N$ a space-like and conformal immersion with zero mean curvature vector. Then F gives its space-like twistor lifts, which are sections of $U_+\big(\bigwedge^2_\pm F^*TN\big)$. Let Q be a complex quartic differential on M defined by F as in the previous paragraph. Then isotropicity of F is given by $Q \equiv 0$, which is equivalent to horizontality of a space-like twistor lift of F ([1]).

Let (N, h) be as in the previous paragraph. The time-like twistor spaces associated with N are fiber bundles in $\bigwedge^2_\pm TN$ such that fibers are hyperboloids of one sheet, and denoted by $U_-\big(\bigwedge^2_\pm TN\big)$. We can refer to [7], [8] for time-like twistor spaces. Let M be a Lorentz surface, which is an analogue of a Riemann surface and a two-dimensional manifold equipped with a holomorphic system of paracomplex coordinate neighborhoods. Let $F : M \longrightarrow N$ be a time-like and conformal immersion with zero mean curvature vector. Then F gives its time-like twistor lifts $\Theta_{F,\pm}$, which are sections of $U_-\big(\bigwedge^2_\pm F^*TN\big)$. Let Q be a paracomplex quartic differential on M defined by F. Then isotropicity of F is given by $Q \equiv 0$. If one of $\Theta_{F,\pm}$ is horizontal, then $Q \equiv 0$ ([1]), while $Q \equiv 0$ does not necessarily mean the horizontality of $\Theta_{F,\pm}$: it is possible that although F is isotropic, the covariant derivatives of $\Theta_{F,\pm}$ are not zero but light-like. The covariant derivatives of $\Theta_{F,\pm}$ are light-like or zero if and only if one of the following holds: (a) the shape operator of a light-like normal vector field vanishes and then Q vanishes; (b) the shape operator of any normal vector field is light-like or zero, and then Q is null or zero ([2]). The conformal Gauss maps of time-like surfaces of Willmore type in 3-dimensional Lorentzian space forms with zero holomorphic quartic differential satisfy Condition (a) ([1]). If N is a 4-dimensional neutral space form, then we can characterize surfaces with Condition (b), based on the Gauss-Codazzi-Ricci equations ([2]).

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Dynamics in nilpotent groups and deformations of locally symmetric rank one manifolds

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We create some analogue of the Sierpiński carpet for nilpotent geometry on horospheres in symmetric rank one negatively curved spaces $H^n_{\mathbb{F}}$ over division algebras $\mathbb{F} \neq \mathbb{R}$, i.e over complex $\mathbb{C}$, quaternionic $\mathbb{H}$, or octonionic/Cayley numbers $\mathbb{O}$. The original Sierpiński carpet in the plane was described by Wacław Sierpiński in 1916 as a fractal generalizing the Cantor set.

Deforming such a Sierpiński carpet with a positive Lebesgue measure at the sphere at infinity $\partial H^n_{\mathbb{F}}$ by its "stretching" compatible with nilpotent geometry, we construct a non-rigid discrete $\mathbb{F}$-hyperbolic groups $G \subset \operatorname{Isom} H^n_{\mathbb{F}}$ whose limit set $\Lambda(G)$ is the whole sphere at infinity $\partial H^n_{\mathbb{F}}$. This answers questions by G.D.Mostow [6], L.Bers [4] and S.L.Krushkal [5] about uniqueness of a conformal or CR structure on the sphere at infinity $\partial H^n_{\mathbb{F}}$ compatible with the action of a discrete isometry group $G \subset \operatorname{Isom} H^n_{\mathbb{F}}$.

Previously, for the real hyperbolic spaces, this problem was solved by Apanasov [1, 2]. Due to D. Sullivan [7] rigidity theorem generalized by Apanasov [2] and [3], Theorem 5.19, the complement of the constructed class of discrete groups $G \subset \operatorname{Isom} H^n_{\mathbb{F}}$ (having a positive Lebesgue measure of the set of vertices of its fundamental polyhedra at infinity) whose limit set $\Lambda(G)$ is the whole sphere at infinity $\partial H^n_{\mathbb{F}}$ consists of groups rigid in the sense of Mostow.

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Characterizing Linear Mappings Through Unital Algebra

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In this paper, we characterize two linear mappings satisfying

$$x, y \in A, x \circ y^* = 0 \Rightarrow 0 = \delta(x) \circ y^* + x \circ \tau(y)^*$$

for all $x, y \in A$, where A be an algebra over a real or complex field K from a unital algebra into its unital bimodule. The structure of linear mappings behaving like Jordan derivations at commutative zero products has been studied extensively. We refer the reader to [1] and [2] for more details.
As is well known, the problem of linear mappings preserving fixed products is a very interesting item in the field of operator algebra. Derivations that can be completely determined by the local action on some subsets of algebra have attracted attention of many researchers. Historically, the study of derivation was initiated during the 1950s and 1960s. Derivations of rings got a tremendous development in 1957, when [3] established two very striking results in the case of prime rings.
We denote by $F(A)$ the subalgebra of A generated by all idempotents in A. Let A be an algebra. An A-bimodule M is said to have the property $\diamond$, if there is an ideal $J \subseteq F(A)$ of A such that $\{m \in M : xmx = 0 \text{ for every } x \in J\} = 0$.

Theorem 1. *Let A be a unital algebra and M be a unital A-bimodule with the property $\diamond$. Suppose that δ is a linear mapping from A into M satisfying*

$$x, y \in A, x \circ y = 0 \Rightarrow \delta(x) \circ y - x \circ \delta(y) = 0$$

and each element of A has a weak inverse. Then A has zero ideal.

Theorem 2. *Let A be a unital algebra and M be a unital A-bimodule with the property $\diamond$. Suppose that δ and τ are linear mappings from A into M satisfying $x, y \in A, x \circ y = 0 \Rightarrow \delta(x) \circ y + x \circ \tau(y) = 0$ and $[A, (\delta - \tau)] = 0$. Then there exists a Jordan derivation $\triangle$ from A into M such that $\triangle(x) = 0$ for every x in A.*

Corollary 3. *Let A be a unital $*$-algebra and M be a unital $*$-A-bimodule with the property $\diamond$. If δ and τ are linear mappings from A into M satisfying*

$$x, y \in A, x \circ y^* = 0 \Rightarrow \delta(x) \circ y^* + x \circ \tau(y)^* = 0,$$

and A is a separating point of M. Then there exist Jordan derivations $\triangle$ and Γ from A into M and $\delta(A) = 0$.

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Edge resolvability and topological characteristics of zero-divisor graphs

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Definition 1. (Zero- Divisor Graph) Zero-divisor graph is a geometric representation of a commutative ring. Zero-divisor graph of ring R is denoted by $\Gamma(R) = (V(\Gamma), E(\Gamma))$, defined by a graph whose vertices are all elements of the zero-divisor set of a ring R, and two distinct vertices z_1 and z_2 are adjacent if and only if $z_1.z_2 = 0$.

Definition 2. (Metric Dimension) Let $G = (V(G), E(G))$ be a graph, and $S \subset V(G)$ be an ordered subset of the principal nodes set, defined as $s = \{\aleph_1, \aleph_2, \aleph_3, ..., \aleph_k\}$. Let $\aleph$ be any principal node in $V(G)$. The identification of a principal node $\aleph$ with respect to S is a k-ordered distance set $\big(d(\aleph, \aleph_1), d(\aleph, \aleph_2), ..., d(\aleph, \aleph_k)\big)$. If each principal node for $V(G)$ has a unique identification according to ordered subset S, then this subset is called resolving set of graph G. The minimum number of elements in the subset S is called the metric dimension of G.

Definition 3. [1] (Edge Metric Dimension) If in a simple and connected graph G, the distinct edges of G have distinct representation with respect to an ordered subset R of vertices of G, then S is known as edge resolving set of G. The minimal edge resolving set of G is called edge metric basis, and its cardinality is called edge metric dimension of G. The edge metric dimension of graph G is denoted by $edim(G)$.

These are some important findings

Theorem 4. [2] *For a graph G, we have*

$$edim(G) = \begin{cases} 1, & \textit{iff } G = P_n, \ \textit{(Path graph)} \\ n-1, & \textit{iff } G = K_n, \ \textit{(Complete graph)} \\ 2, & \textit{if } G = C_n, \ \textit{(Cycle graph)} \\ n-2, & \textit{if } G \cong K_{1,n} \ \textit{(except } K_{1,1}), \ \textit{or a bipartite graph} \end{cases}$$

Theorem 5. [3] *The diameter of* $\Gamma(R) \leq 3$, *where R is a commutative ring.*

Theorem 6. *The edge metric dimension of the zero-divisor graph of R is finite iff R is finite, where $R - \{0\}$ is a commutative ring but not an integral domain.*

Theorem 7. *For the ring $\mathbb{Z}_m$, where $m \geq 1$, we have*

$$edim\left(\Gamma(\mathbb{Z}_m)\right) = \begin{cases} undefined, & if\ m = p\ is\ a\ prime \\ p-2, & if\ m = p^2 and\ p > 2 \end{cases}$$

Theorem 8. *Consider the ring $\mathbb{Z}_m$, where $m \geq 1$, we have*

$$edim\left(\Gamma(\mathbb{Z}_m)\right) = \begin{cases} undefined, & if\ m = 2p, where\ p\ is\ an\ even\ prime \\ p-2, & if\ m = 2p, where\ p\ is\ an\ odd\ prime \end{cases}$$

Theorem 9. *Consider the ring $\mathbb{Z}_m[i]$, where p is a prime then*

$$edim\left(\Gamma(\mathbb{Z}_m[i])\right) = \begin{cases} undefined, & if\ m = 2 \\ p^2-2, & if\ m = p^2 \end{cases}$$

Theorem 10. *Consider the ring $\mathbb{Z}_p[i]$, where p is a prime. If $p \equiv m(mod\ 4)$ then*

$$edim\left(\Gamma(\mathbb{Z}_p[i])\right) = \begin{cases} 2p-4, & if\ m = 1 \\ undefined, & if\ m = 2 \\ undefined, & if\ m = 3 \end{cases}$$

Theorem 11. *Consider the ring $\mathbb{Z}_m[i]$, then Zagreb first index (M_1)*

$$M_1\left(\Gamma(\mathbb{Z}_m[i])\right) = \begin{cases} 1, & if\ m = 2 \\ (p^2-2)^2(p^2-1), & if\ m = p^2 where\ p\ is\ a\ prime \end{cases}$$

Theorem 12. *Consider the ring $\mathbb{Z}_p[i]$, where p is a prime. If $p \equiv m(mod\ 4)$ then*

$$M_1\left(\Gamma(\mathbb{Z}_p[i])\right) = \begin{cases} (2p-2)(p-1)^2, & if\ m = 1 \\ undefined, & if\ m = 2 \\ 0, & if\ m = 3 \end{cases}$$

Remark 13. This article examines the edge metric dimension and topological nature of $\Gamma(R)$. We have looked closely at edge metric dimension of integers modulo m, and Gaussian integers modulo m. We also discovered the first Zagreb index, second Zagreb index, and Sombor index of the zero divisor graph of the Gaussian integers modulo m. These findings are helpful for researching the structural characteristics of rings and chemical compounds.

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From minimality to maximality via metric reflection

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In 1934 Đuro Kurepa [4] introduced the pseudometric spaces which, unlike metric spaces, allow the zero distance between different points.

Definition 1. Let X be a set and let $d\colon X^2 \to \mathbb{R}$ be a non-negative, symmetric function such that $d(x,x) = 0$ for every $x \in X$. The function d is a *pseudometric* on X if it satisfies the triangle inequality.

If d is a pseudometric on X, we say that (X,d) is a *pseudometric space*.

Definition 2 ([3]). Let (X,d) and (Y,ρ) be pseudometric spaces. The spaces (X,d) and (Y,ρ) are *combinatorially similar* if there exist bijections $\Psi\colon Y \to X$ and $f\colon d(X^2) \to \rho(Y^2)$ such that $\rho(x,y) = f\big(d(\Psi(x),\Psi(y))\big)$ for all $x, y \in Y$. In this case, we will say that $\Psi\colon Y \to X$ is a *combinatorial similarity* and that (X,d) and (Y,ρ) are combinatorially similar pseudometric spaces.

Definition 3. Let (X,d) be a pseudometric space. A bijection $f : X \to X$ is a *pseudoidentity* if the equality $d(x,f(x)) = 0$ holds for every $x \in X$.

The groups of all combinatorial self-similarities and all pseudoidentities of a pseudometric space (X,d) will be denoted by $\mathbf{Cs}(X,d)$ and $\mathbf{PI}(X,d)$ respectively. Thus, for every pseudometric space (X,d) we have $\mathbf{PI}(X,d) \subseteq \mathbf{Cs}(X,d) \subseteq \mathbf{Sym}(X)$, where $\mathbf{Sym}(X)$ is the symmetric group of all permutations of the set X.

For every nonempty pseudometric space (X,d), we define a binary relation $\overset{0(d)}{=}$ on X by

$$(x \overset{0(d)}{=} y) \Leftrightarrow (d(x,y)=0), \quad \text{for all } x, y \in X.$$

Proposition 4. *Let X be a nonempty set and let $d\colon X^2 \to \mathbb{R}$ be a pseudometric on X. Then $\overset{0(d)}{=}$ is an equivalence relation on X and, in addition, the function δ_d,*

$$\delta_d(\alpha,\beta) := d(x,y), \quad x \in \alpha \in X/\overset{0(d)}{=}, \quad y \in \beta \in X/\overset{0(d)}{=},$$

is a correctly defined metric on the quotient set $X/\overset{0(d)}{=}$.

In what follows we will say that the metric space $(X/\overset{0(d)}{=}, \delta_d)$ is the *metric reflection* of (X,d).

Let us define a class $\mathcal{IP}$ of pseudometric spaces as follows.

Definition 5. A pseudometric space (X,d) belongs $\mathcal{IP}$ if the equalities

$$\mathbf{Cs}(X,d) = \mathbf{PI}(X,d) \quad \text{and} \quad \mathbf{Cs}(X/\overset{0(d)}{=}, \delta_d) = \mathbf{Sym}(X/\overset{0(d)}{=}) \quad \text{hold.}$$

Our main goal is to describe the structure of pseudometric spaces belonging to $\mathcal{IP}$. To do this, we introduce into consideration pseudometric generalizations of some well-known classes of metric spaces.

Let (X, d) be a metric space. Recall that the metric d is said to be *strongly rigid* if, for all x, y, u, $v \in X$, the condition $d(x, y) = d(u, v) \neq 0$ implies ($x = u$ and $y = v$) or ($x = v$ and $y = u$). The *discrete metric* d on X is defined by $d(x, y) = k$, if $x \neq y$ and $d(x, y) = 0$, if $x = y$ for any $x, y \in X$ and arbitrary fixed $k > 0$.

Definition 6. Let (X, d) be a pseudometric space. Then d is *discrete* (*strongly rigid*) if all metric subspaces of (X, d) are discrete (strongly rigid).

Definition 7. A pseudometric space (X, d) is a *pseudorectangle* if all three-point metric subspaces of (X, d) are strongly rigid and isometric and, in addition, there is a four-point metric subspace Y of (X, d) such that for every $x \in X$ we can find $y \in Y$ satisfying $d(x, y) = 0$.

Let X be a nonempty set and $P = \{X_j : j \in J\}$ be a set of nonempty subsets of X. The set P is a *partition* of X with the *blocks* X_j, $j \in J$, if $\cup_{j \in J} X_j = X$ and $X_{j_1} \cap X_{j_2} = \varnothing$ for all distinct j_1, $j_2 \in J$.

Now we are ready to characterize the pseudometric spaces satisfying equality

$$\mathbf{Cs}(X/\overset{0(d)}{=}, \delta_d) = \mathbf{Sym}(X/\overset{0(d)}{=}) \quad \text{(see [1] for more details).}$$

The following theorem is, in fact, a pseudometric modification of the main result of [2].

Theorem 8. *Let (X, d) be a nonempty pseudometric space. Then the following statements are equivalent:*

- (i) *At least one of the following conditions has been fulfilled:*
  - (i_1) *(X, d) is strongly rigid;*
  - (i_2) *(X, d) is discrete;*
  - (i_3) *(X, d) is a pseudorectangle.*
- (ii) *The equality*

$$\mathbf{Cs}(X/\overset{0(d)}{=}, \delta_d) = \mathbf{Sym}(X/\overset{0(d)}{=})$$

holds.

The next theorem can be considered as one of the main results of our work.

Theorem 9. *Let (X, d) be a nonempty pseudometric space and let $\{X_j : j \in J\}$ be a partition of X corresponding the equivalence relation $\overset{0(d)}{=}$. Then $(X, d) \in \mathcal{IP}$ if and only if*

$$|X_{j_1}| \neq |X_{j_2}|$$

holds whenever $j_1, j_2 \in J$ are distinct and, in addition, at least one of the following conditions has been fulfilled:

- (i) *(X, d) is strongly rigid;*
- (ii) *(X, d) is discrete;*
- (iii) *(X, d) is a pseudorectangle.*

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Thurston norm and Euler classes of bounded mean curvature foliations on hyperbolic 3-Manifolds

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Let M be a closed, oriented 3-manifold, and suppose that M contains no non-separating 2 - spheres or tori. For example, M is a closed oriented hyperbolic 3-Manifold.

The Thurston norm on $H_2(M,\mathbb{Z})$ is defined as follows ([1]):

$$||a||_{Th} = inf\{\chi_-(\Sigma)|\ \Sigma \text{ is an embedded oriented surface representing } a \in H_2(M,\mathbb{Z})\}, \quad (1)$$

where $\chi_-(\Sigma) = max\{-\chi(\Sigma), 0\}$. Recall that $\chi(\Sigma) = 2-2g$ denotes the Euler characteristic of a surface Σ of genus g. When Σ is not connected, define $\chi_-(\Sigma)$ to be the sum $\chi_-(\Sigma_1)+\cdots+\chi_-(\Sigma_k)$, where $\Sigma_i,\ i = 1,\ldots,k$ are the connected components of Σ. As Thurston showed, the Thurston norm can be extended in a unique way to the norm in $H_2(M,\mathbb{R})$.

The dual Thurston norm can be defined on $H^2(M,\mathbb{R})$ by the formula

$$||\alpha||^*_{Th} = \sup_{\Sigma} \frac{<\alpha,[\Sigma]>}{2g(\Sigma)-2}, \quad (2)$$

where $\alpha \in H^2(M,\mathbb{R})$ and the supremum being taken over all connected, oriented surfaces Σ embedded in M whose genus g is at least 2.

Recall that a *taut* foliation is a codimension one foliation of a closed manifold with the property that every leaf meets a transverse circle. Equivalently, by a result of Dennis Sullivan [2], a codimension one foliation is taut if there exists a Riemannian metric that makes each leaf a minimal surface. Thurston proved that the convex hull of the Euler classes of taut foliations on M is the unit ball for the dual Thurston norm. In particular, the Thurston norm $||e(\mathcal{F})||^*_{Th}$ of the Euler class $e(\mathcal{F}) \in H^2(M,\mathbb{R})$ of a taut foliation $\mathcal{F}$ is no more then one.

We represent the following result.

Theorem 1. *Let M be a closed oriented hyperbolic 3-Manifold and $\mathcal{F}$ be a two-dimensional transversely oriented foliation $\mathcal{F}$ whose leaves have the modulus of mean curvature bounded above by the fixed positive constant H_0. Then*

– *If $H_0 \leq 1$, we have $\mathcal{F}$ is taut and $||e(\mathcal{F})||^*_{Th} = 1$.*
– *If $H_0 > 1$, we have*

$$||e(\mathcal{F})||^*_{Th} \leq 2\pi \frac{1600 H_0^2 Vol(M)^2}{C_0^3 inj(M)} + \frac{300 Vol(M)}{inj(M)} + 1,$$

where $C_0 = 2\min\{inj(M), (\coth)^{-1}(H_0)\}$, $Vol(M)$ *is the volume of* M *and* $inj(M)$ *is the injectivity radius of* M.

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Nijenhuis geometry and its applications

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This presentation is focused on some results of the long-term research programme *Nijenhuis Geometry* initiated several years ago in cooperation with Vladimir Matveev and Andrey Konyaev.

A *Nijenhuis operator* $L = \big(L^i_j(x)\big)$ is defined to be a field of endomorphisms on a smooth manifold M such that its Nijenhuis torsion identically vanishes, i.e.,

$$\mathcal{N}_L(\xi, w) = L^2[\xi, \eta] + [L\xi, L\eta] - L[L\xi, \eta] - L[\xi, L\eta] = 0, \tag{1}$$

for arbitrary vector fields ξ, η on M. The pair (M, L) is called a *Nijenhuis manifold*.

Relation (1) is the simplest differential-geometric condition on a field of endomorphisms, and that is the reason why Nijenhuis operators appear in many areas of differential geometry and mathematical physics. In the theory of integrable bi-Hamiltonian systems, they serve as recursion operators and their role in this area has been well understood for many years due to pioneering works by F. Magri, Y. Kosmann-Schwarzbach and F. Turiel. A classical fact in complex geometry is that an almost complex structure is integrable if and only if it is Nijenhuis (Newlander–Nireberg theorem). In the context of metric projective geometry, Nijenhuis operators played a crucial role in various classification problems (AB and V. Matveev). They naturally occur in the study of infinite dimensional Poisson brackets of hydrodynamic type (E. Ferapontov *et al*). Even in algebra, Nijenhuis operators turns out to be useful in the theory of integrable systems on Lie algebras and Lie pencils (A. Panasyuk), and also appear as left symmetric algebras.

Besides various applications, our motivation is as follows. Classical geometries are defined by means of a tensor of order 2. For Riemannian, sub-Riemannian, symplectic and Poisson structures, this tensor is a bilinear form (co- or contravariant, symmetric or skew-symmetric). In this list, one type of tensors is still missing: linear operators. Nijenhuis geometry would be a very natural candidate to fill this gap.

Thus, *Nijenhuis Geometry* research programme is aimed at systematic development of the theory of Nijenhuis manifolds. Our vision and first results are presented in [1–8]. More specifically, our goal is to *re-direct the research agenda* in this area from *tensor analysis at generic points* to studying *singularities and global properties*. The ultimate goal of our research programme is to answer three fundamental questions:

(A) **Local description:** to what form can one bring a Nijenhuis operator near almost every point by a local coordinate change?

(B) **Singular points:** what does it mean for a point to be generic or singular in the context of Nijenhuis geometry? What singularities are non-degenerate/stable? How do Nijenhuis operators behave near non-degenerate and stable singular points?

(C) **Global properties:** what restrictions on a Nijenhuis operator are imposed by the topology of the underlying manifold? And conversely, what are topological obstructions to a Nijenhuis manifold carrying a Nijenhuis operator with specific properties?

Below are some of our easy-to-formulate results in the area.

Theorem 1. *Let L be a Nijenhuis operator and $\sigma_1, \dots, \sigma_n$ be the coefficients of its characteristic polynomial $\chi(t) = \det(t \cdot \mathrm{Id} - L) = t^n - \sum_{k=1}^{n} \sigma_k\, t^{n-k}$. Then in any local coordinate system $x_1, \dots, x_n$ the following matrix relation hold:*

$$J(x)\, L(x) = S_\chi(x)\, J(x), \quad \textit{where } S_\chi(x) = \begin{pmatrix} \sigma_1(x) & 1 & & \\ \vdots & 0 & \ddots & \\ \sigma_{n-1}(x) & \vdots & \ddots & 1 \\ \sigma_n(x) & 0 & \dots & 0 \end{pmatrix} \tag{2}$$

and $J(x)$ is the Jacobi matrix of the collection of functions $\sigma_1, \dots, \sigma_n$ w.r.t. the variables $x_1, \dots, x_n$.

Theorem 2. *Let L be a real-analytic Nijenhuis operator of the form*

$$L(x) = L_{\mathrm{lin}}(x) + R(x), \quad \textit{where } L_{\mathrm{lin}}(x) = \mathrm{diag}(x_1, x_2, \dots, x_n)$$

and $R(x)$ denotes a non-linear perturbation (of order ≥ 2). Then $L(x)$ is linearisable, i.e., there exists a real analytic change of variables $x \mapsto y$ such that in the new coordinates $L(y) = \mathrm{diag}(y_1, y_2, \dots, y_n)$.

Theorem 3. *A Nijenhuis operator on a closed connected manifold cannot have non-constant complex eigenvalues.*

Theorem 4. *Consider a real analytic* gl*-regular Nijenhuis operator L (*gl*-regularity means that each eigenvalue of L may have arbitarary multiplicity but only one linearly independent eigenvector). Then there exist local coordinate systems $u = (u^1, \dots, u^n)$ and $v = (v^1, \dots, v^n)$ in which L reduces to the first and second companion forms:*

$$L(u) = L_{\mathsf{comp1}} = \begin{pmatrix} \sigma_1 & 1 & & \\ \vdots & 0 & \ddots & \\ \sigma_{n-1} & \vdots & \ddots & 1 \\ \sigma_n & 0 & \dots & 0 \end{pmatrix} \quad \textit{and} \quad L(v) = L_{\mathsf{comp2}} = \begin{pmatrix} 0 & 1 & & \\ \vdots & \ddots & \ddots & \\ 0 & \dots & 0 & 1 \\ \sigma_n & \sigma_{n-1} & \dots & \sigma_1 \end{pmatrix},$$

where σ_i are the coefficients of the characteristic polynomial of L in the corresponding coordinate system.

Theorem 5. *Let M^2 be either a sphere or a closed Riemann surface of genus ≥ 2. Then M^2 cannot carry any* gl*-regular Nijenhuis operator L except for $L = \alpha \,\mathrm{Id} + \beta A$, where A is a complex structure on M^2 and $\alpha, \beta \in \mathbb{R}$, $\beta \neq 0$. A non-orientable closed 2-manifold different from a Klein bottle cannot carry any* gl*-regular Nijenhuis operator.*

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Shape optimization in the batch crystallization of CAM

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The citric acid monohydrate (CAM) is an important organic substance but, until 1997, the scientific literature covered mostly the kinetics of nucleation [4] and the crystal growth [5] rather than its production via the crystallization by cooling in a stirred tank reactor (STR). The Department of Chemical Engineering at the University "La Sapienza" of Rome decided to fill that sci-tech gap through a meticulous investigation, with three STRs at the laboratories of San Pietro in Vincoli's district, on the crystallization in discontinuous (batch) of CAM from aqueous solutions. The author participated in that cutting edge experience, as experimenter and coder under the supervision of Prof. Barbara Mazzarotta, in the years 1997-1998 [1]. Our specific tasks were to spot the main operating conditions, to modify them until an *optimal* crystal size distribution (CSD), i.e., large-sized homogeneous crystals of CAM, and to write a QBasic program predicting the outcomes of any test in batch reactors [2]. Here we focus on the influence of the STRs' geometry, i.e., the role played by the tanks in crystallizing the CAM thanks to their differently shaped bottoms (flat, hemispherical, conical). All the data, collected and simulated, show that the round-bottomed crystallizer gives the best CSD, performing better than the conical-bottomed STR, and that we should discard the flat-bottomed STR for the poor quality of its crystalline product [3]. The homogenous distribution of large crystals from the round-bottomed STR is due to the *optimal* suspension state that such shape provides for the dispersed phase of CAM particles [6], as confirmed by the computational fluid-dynamics software `VisiMix`.

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On classification of almost positive posets

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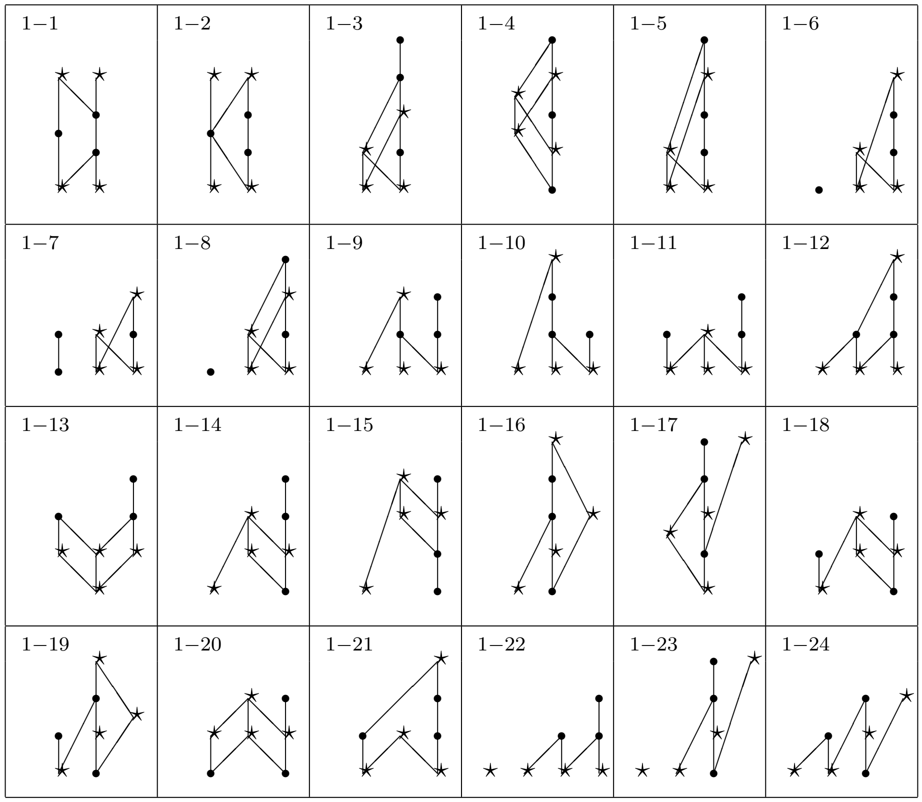
We identify finite posets with the Hasse diagrams and often use by the default terms for a poset S based on the analogous terms for its quadratic Tits form $q_S(z)$ (for example, "positive poset" means that $q_S(z)$ is positive). We call a non-positive poset S *almost positive* if there exists $x \in S$ (called *special*) such that $S \setminus x$ is positive (all the positive poset are described, by the method of minimax equivalence [1], in [2]). A special case of such posets are P-critical ones when $S \setminus x$ is positive for any $x \in S$ (they are described also in [2]). We have proved the next theorem.

Theorem 1. *For a non-negative poset S of order n the following conditions are equivalent: (1) S is almost positive; (2) the subgroup $\{t \in \mathbb{Z}^{n+1} \mid q_S(t) = 0\}$ of $\mathbb{Z}^{n+1}$ is infinite cyclic.*

It follows from this theorem that the classification of the serial almost positive non-negative posets (which include all ones of order $n > 8$) is given by Theorems 3, 4 [3] and non-serial ones of order $n = 6, 7, 8$ by calculations using a computer program [4]. We study the second case with the help of our method of minimax equivalence. In particular, in the case $n = 7$, after elimination of the P-critical posets [2], we have the following classification up to isomorphism and dyality (all posets of each table are minimax equivalent; the symbols $\star$ denote special points).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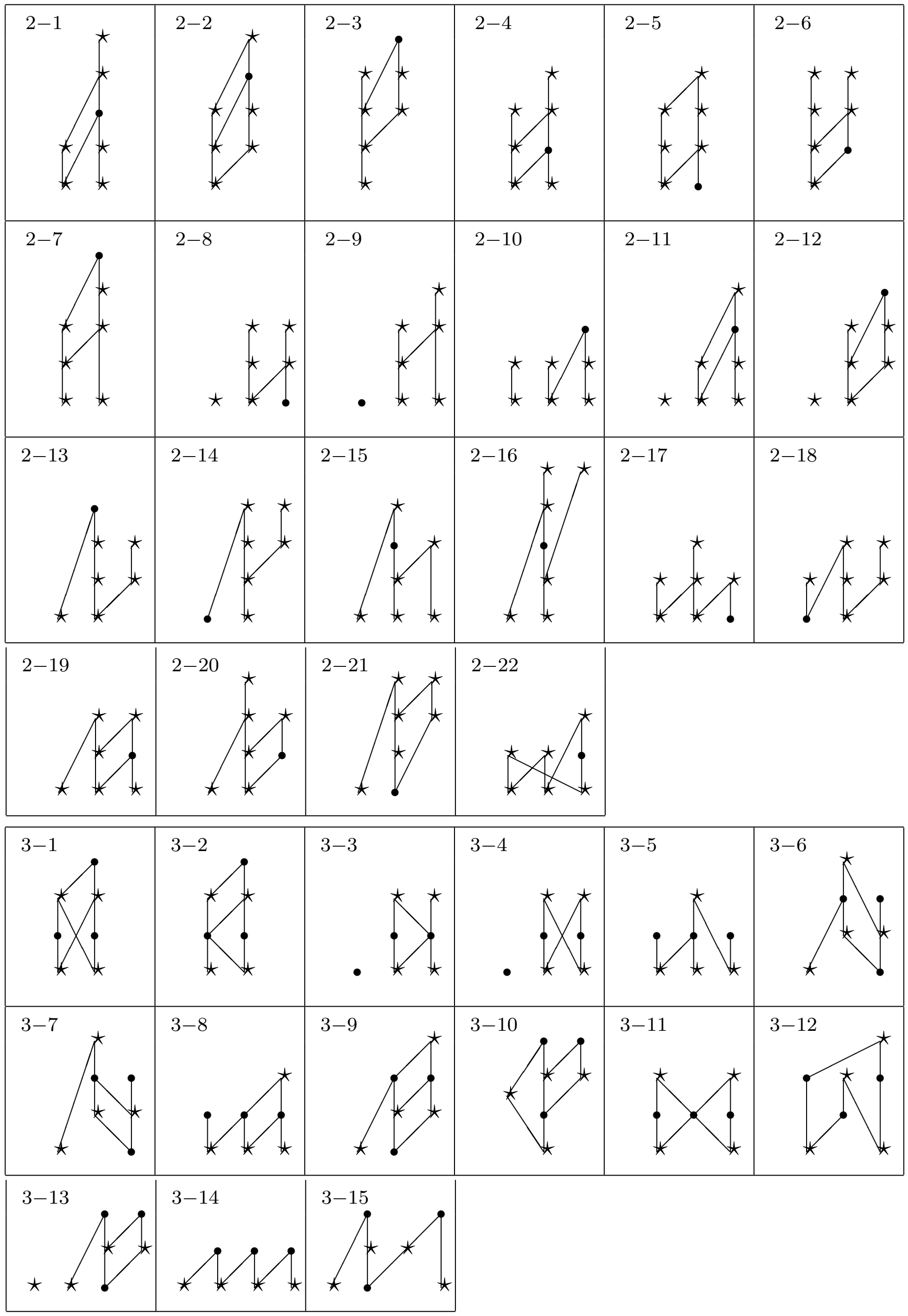

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Homotopies to Diffeomorphisms in Symplectic Field Theory

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Homotopies between non-compact Lagrangian submanifolds are considered, and using the Fukaya conjecture relative to the Witten deformation of higher product structures conforming a Fukaya category $\mathcal{W}(H)$, from the perspective of the Floer complexes, which determine diffeomorphisms $C_{-*}(\Omega_x) \to \mathcal{W}(H)$, whose space of paths go from $\gamma(x)$, to $\phi(x)$, foreseen in $HW^*(L_0, L_1) \cong H_{-*}(\mathcal{P}_{x_0,x_1})$. Then the field ramification of the space $C_{-*}(\Omega_x)$, is a connection obtained under the following commutative category scheme [1]:

$$
\begin{array}{ccccc}
mod(B) & & \xrightarrow{\mathcal{R}^{-1}} & & C \\
\nearrow \quad \downarrow & & & & \nearrow \ \downarrow \\
O_c(\phi) \in H(\mathrm{mod} f(C_{-*}(\Omega Z)) & & \longrightarrow & H(\mathcal{M}) & \mathcal{M} \\
\downarrow \quad \nearrow \Omega Z & & \rightarrow & \downarrow \mathrm{embb} & \nearrow \\
C_{-*}(\Omega_x) & & \xrightarrow{Diff} & \mathcal{W}(H) \ni \phi &
\end{array}
\tag{1}
$$

Note. Here $\mathcal{W}(H)$, represents the wrappings of the flow of geodesics, which physically represents that happen in the dual space obtained for the product of the diffeomorphism given in the Čech complex defined by $C = \oplus_I \Gamma(U_I)[-d]$, that is to say, of the "states" $\phi(x)$, which are connected by the paths of the cohomology of the paths in Z, from $\phi(x_0)$, to $\phi(x_1)$. the other conjecture that must be planted is that as consequence of the derived categories scheme(1) is:

Conjecture 1. *Direction is time and translation is space in the space-time.*

Keywords: Čech Complex, Diffeomorphisms, Floer Cohomology, Fukaya Category, Homotopy, Lagrangian submanifolds.

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Algebraic and geometric methods in Relativistic Quantum Mechanics and Schwartz distribution spaces defined on Minkowski space-time

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The intent of the general analysis conducted during this talk is to start from Laurent Schwartz distribution spaces, based also on the Minkowski space-time, following the spirit of Linear Algebra and Geometry, and offer precise meanings and rigorous support to many calculus methods of Quantum Mechanics.

Our approach not only provides a rigorous and efficient justification for the use of many quantum mechanics mathematical tools substantially as they usually show up in the physical practice, but - by a "correct" formulation of the calculus methods in terms of contemporary mathematics - it helps to reach a deeper comprehension of the physical structures studied in Quantum Mechanics.

In particular, in this direction, we consider a new definition of state spaces for quantum systems. These structures are often identified with separable Hilbert spaces, leading immediately to the so called "non-normalizable" states - which revealed fundamental in the development of the quantum mechanics. Indeed, those particular states should be normalizable, but with respect other convenient scalar products, orthogonal to the initial one.

Some researchers in theoretical quantum mechanics already understand that the state space of a Quantum system should be a larger structure than Hilbert spaces, sometimes they call it "physical Hilbert space", without introducing a clear definition. We have already shown in the past ([1, 2, 3]) that "physical Hilbert spaces" can be smoothly identified with distribution spaces on suitable Euclidean spaces, depending from the nature of the quantum system considered. Such distribution spaces should be endowed with some algebraic-topological structures, such as the operations of continuous superposition and extended Dirac products.

In particular, the extended Dirac product allows us to introduce new usual scalar products upon some distinguished subspaces of distribution spaces, endowing those subspaces with non-separable Hilbert structures, which clarify definitely the role of the so-called non-normalizable states. For example, the singular Dirac distributions and the celebrated De Broglie waves become elements of those new non-separable Hilbert spaces and, consequently, they acquire the status of normalizable states, as it seems completely natural because of the physical usual probability interpretation of such states. The new operation of continuous-superposition revealed the right tool which allows us to build - in a mathematically rigorous way - the extended Linear Algebra of Dirac, in distribution spaces, using the Schwartz natural topological-linear structures of those spaces.

More precisely, we saw that the natural algebraic-topological structure of those spaces allows to define an extension of the finite linear combination, when the sets indexing the families of vectors are continuous sets, even in the case in which the systems of coefficients show a continuous-infinity of terms different from zero. Now, it appears utterly clear how our approach to distribution theory and QM induces a renewed geometrical vision of the functional analysis developments of the two theories themselves: we realize the existence of continuous bases of distributions and corresponding coordinate systems, continuous matrices of continuous operators, tangent and cotangent spaces of infinite dimensional manifolds, modeled upon distribution spaces, endowed with suitable and comfortable continuous bases, leading to a infinite dimensional differential geometry theory much closer to the finite dimensional one.

As a possible application, we solve the problem of quantizing the relativistic Hamiltonian of a free massive particle massive particle (rest mass different from 0). In distribution state spaces, we find a natural way to define the relativistic Hamiltonian operator and its associated Schrödinger equation. We, then, deduce the equivalent continuity equation for the Born probability density and study some its different (but equivalent) expressions. We determine the possible probability currents and flux velocity fields associated with the particle-field evolution.

The principal properties of the Hamiltonian operator are presented in the following theorem.

Theorem 1. *The relativistic Hamiltonian operator $\hat{H}$ reveals the unique linear continuous operator on $\mathcal{S}'(\mathbb{M}_4)$ sending each de Broglie wave β_p to the tempered distribution $H_{\boldsymbol{p}}\beta_p$, where $\boldsymbol{p}$ denotes the spatial part of p. Moreover, we see that:*

- *$\hat{H}$ reveals Schwartz diagonalizable (Schwartz non-defective): there exists a Schwartz basis of $\mathcal{S}'(\mathbb{M}_4)$ constituted by eigenvectors of $\hat{H}$;*
- *the operator $\hat{H}$ reveals regular and Hermitian in the Schwartz sense: it could be restricted to an endomorphism of the test function space $\mathcal{S}(\mathbb{M}_4)$ and its restriction reveals Hermitian with respect to the standard Dirac inner product of $\mathcal{S}(\mathbb{M}_4)$.*

In non-relativistic QM, the evolution equation takes also the time-dependent form

$$\mathcal{E}'_{\mathbf{H}} \ : \ \imath\hbar\,\psi'(t) = \hat{\mathbf{H}}\psi(t),$$

with

$$\psi : \mathbb{T} \to \mathcal{S}'(\mathbb{X}_3),$$

smooth curve parametrized by time. We desire to find an analogous expression in the relativistic case.

Theorem 2. *Let us fix any $\psi_0 \in \mathcal{S}'(\mathbb{P}_3)$. Set now*

$$\psi(t) = e^{-(\imath/\hbar)t\,\hat{\mathbf{H}}}\,\psi_0,$$

for every time t. Here, as usual in Schwartz linear algebra, we define

$$e^{-(\imath/\hbar)t\,\hat{\mathbf{H}}}\,\psi_0 := \int_{\mathbb{P}_3} e^{-(\imath/\hbar)t\,\mathbf{H}}\,(\psi_0)_\eta\,\eta,$$

for every tempered wave ψ_0 defined upon $\mathbb{X}_3$. Then, the above curve ψ verifies the Schrödinger equation $\mathcal{E}_{\mathbf{H}}$, that is, it fulfills the relation

$$\mathcal{E}_{\mathbf{H}} \ : \ \imath\hbar\,\psi'(t) = \hat{\mathbf{H}}\psi(t),$$

for every time t.

We moreover prove the following theorem.

Theorem 3. *Any solution κ of the Schrodinger equation $\mathcal{E}$ determines a distribution-curve*

$$\psi : \mathcal{S}(\mathbb{T}) \to \mathcal{S}'(\mathbb{X}_3)$$

defined by

$$< \psi(\phi_0), \phi > = < \kappa, \phi_0 \otimes \phi > .$$

Viceversa, any distribution curve, satisfying $\mathcal{E}'$, determines, by the Schwartz kernel theorem, a solution of the Shrodinger equation $\mathcal{E}$.

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Hurwitz Zeta Functions and Ramanujan's Identity for Odd Zeta Values

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One of the famous identities given by Ramanujan which has attracted the attention of several mathematicians over the years is the following intriguing identity involving the odd values of the Riemann zeta function:

Theorem 1 (Ramanujan's formula for $\zeta(2n+1)$)**.** *If α and β are positive real numbers such that $\alpha\beta = \pi^2$ and if $n \in \mathbb{Z} \setminus \{0\}$, then we have*

$$\alpha^{-n}\left\{\frac{1}{2}\,\zeta(2n+1)+\sum_{m=1}^{\infty}\frac{m^{-2n-1}}{e^{2\alpha m}-1}\right\}-(-\beta)^{-n}\left\{\frac{1}{2}\,\zeta(2n+1)+\sum_{m=1}^{\infty}\frac{m^{-2n-1}}{e^{2\beta m}-1}\right\}$$
$$=2^{2n}\sum_{k=0}^{n+1}\frac{(-1)^{k-1}B_{2k}\,B_{2n-2k+2}}{(2k)!\,(2n-2k+2)!}\,\alpha^{n-k+1}\beta^{k}. \qquad (1)$$

where B_n denotes the n-th Bernoulli number.

Theorem 1 appears as Entry 21 in Chapter 14 of Ramanujan's second notebook [1, 173]. Notice that the function $\frac{1}{e^{2\pi x}-1}$ appears in several of Ramanujan's identities and has the following integral representation:

$$\frac{1}{e^{2\pi x}-1}=\frac{1}{2i\pi}\int_{(c)}\frac{\zeta\left(1-s\right)}{2\cos\left(\frac{\pi s}{2}\right)}x^{-s}\mathrm{d}s,$$

where (c) denotes the vertical line $\Re(s) = c$ with c an arbitrary real number such that $1 < c < 2$. We generalize this function and define the *Hurwitz kernel* by

$$\Psi\left(x,a;k\right):=\frac{1}{2i\pi}\int_{(c)}\frac{\zeta\left(1-s,a\right)}{2k\cos\left(\frac{\pi(s+k-1)}{2k}\right)}x^{-s}\mathrm{d}s$$

$$= \frac{2a-1}{2\pi x} - \frac{1}{2k\cos\left(\frac{\pi(k-1)}{2k}\right)} + \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{x^{2k-1}}{x^{2k}+(n+a)^{2k}}, \tag{2}$$

where $\zeta(s,a)$ is the Hurwitz zeta function. Let $\Psi_\alpha(x,a;k) = \Psi\left(\frac{\alpha x}{\pi}, a; k\right)$. We now find several identities involving this kernel which generalize Ramanujan's identity such as the following:

Theorem 2. *Let $\alpha, \beta \in \mathbb{R}^+$ such that $\alpha\beta = \pi^2$ and let $k, N \in \mathbb{N}$. Then, we have*

$$\beta^{k(N+1)-1}\left(\sum_{n=0}^{\infty}\frac{\Psi_\alpha(n+b,a;k)}{(n+b)^{2k(N+1)-1}} + \frac{\zeta(2k(N+1)-1,b)}{2k\cos\left(\frac{\pi(k-1)}{2k}\right)}\right)$$
$$= (-1)^N\alpha^{k(N+1)-1}\left(\sum_{n=0}^{\infty}\frac{\Psi_\beta(n+a,b;k)}{(n+a)^{2k(N+1)-1}} + \frac{\zeta(2k(N+1)-1,a)}{2k\cos\left(\frac{\pi(k-1)}{2k}\right)}\right)$$
$$+\sum_{p=0}^{N+1}(-1)^{p+1}\zeta(2kp,a)\,\zeta(2k(N-p+1),b)\,\alpha^{kp-1}\beta^{k(N+1-p)-1}. \tag{3}$$

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Global asymptotic stability of generalized homogeneous dynamical systems

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The equivalence between uniform asymptotic stability and exponential stability for generalized homogeneous non-autonomous differential equations

$$x' = f(t,x) \tag{1}$$

is established. This results we prove in the framework of general non-autonomous (cocycle) dynamical systems.

Let $\mathbb{R} := (-\infty, +\infty)$ and $C(\mathbb{R}\times\mathbb{R}^n, \mathbb{R}^n)$ be the space of all continuous functions $f : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ equipped with the compact-open topology. Denote by $(C(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^n), \mathbb{R}, \sigma)$ the shift dynamical system on $C(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^n)$, i.e., $\sigma(\tau, f) := f^\tau$ and $f^\tau(t,x) := f(t+\tau,x)$ for any $t,\tau \in \mathbb{R}$ and $x \in \mathbb{R}^n$.

Along with equation (1) we consider its *H-class* [4, 2, 6, 10], i.e., the family of equations

$$v' = g(t,v), \tag{2}$$

where $g \in H(f) := \overline{\{f^\tau |\ \tau \in \mathbb{R}\}}$, $f^\tau(t,u) = f(t+\tau,u)$ for any $(t,u) \in \mathbb{R}\times\mathbb{R}^n$ and by bar we denote the closure in $C(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^n)$. We will suppose also that the function f is *regular* [9, ChIV], i.e., for every equation (2) the conditions of existence, uniqueness and extendability on $\mathbb{R}_+$ are fulfilled. Denote by $\varphi(t,v,g)$ the solution of equation (2), passing through the point $v \in \mathbb{R}^n$ at the initial moment $t = 0$.

Let $\mathbb{R}^n$ with euclidian norm $|x| := \sqrt{x_1^2 + \ldots + x_n^2}$. Denote by

$$|x|_{r,p} := \left(\Sigma_{i=1}^n |x_i|^{\frac{p}{r_i}}\right)^{\frac{1}{p}}, \tag{3}$$

where $r := (r_1, \ldots, r_n)$, $r_i > 0$ for any $i = 1, \ldots, n$ and $p \geq 1$. Denote by $\rho(x) := |x|_{r,p}$ and $\Lambda_\varepsilon^r := diag(\varepsilon^{r_i})_{i=1}^n$.

Definition 1. A function $f \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ is said to be:

(1) r-homogeneous ($r \in (0, +\infty)^n$) of degree $m \in \mathbb{R}$ [7, 11] if $f(t, \Lambda_\varepsilon^r x) = \varepsilon^m \Lambda_\varepsilon^r f(t, x)$ for any $\varepsilon > 0$ and $(t, x) \in \mathbb{R} \times \mathbb{R}^n$;
(2) Lagrange stable [4] if the set $H(f)$ is compact in $C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$.

Remark 2. If the function $f \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ is r homogeneous of degree $m \geq 0$, then $f(t, 0) = 0$ for any $t \in \mathbb{R}$.

Definition 3. The trivial solution of equation (1) is said to be:

(1) *uniformly stable*, if for all positive number ε there exists a number $\delta = \delta(\varepsilon)$ ($\delta \in (0, \varepsilon)$) such that $|x| < \delta$ implies $|\varphi(t, x, f^\tau)| < \varepsilon$ for all $t, \tau \in \mathbb{R}_+$;
(2) *attracting (respectively, uniformly attracting)*, if there exists a positive number a

$$\lim_{t \to +\infty} |\varphi(t, x, f^\tau)| = 0$$

for all (respectively, uniformly with respect to) $|x| \leq a$ and $\tau \in \mathbb{R}_+$;
(3) *asymptotically stable (respectively, uniformly asymptotically stable*, if it is uniformly stable and attracting (respectively, uniformly attracting).

Remark 4. 1. Note that from the results given in the works [1],[9] it follows the equivalence of standard definition of uniform stability (respectively, global uniform asymptotically stability) and of the one given above for the equation (1) with regular right hand side.

2. From the results of G. Sell [9] it follows that for the differential equations (1) with the regular and Lagrange stable right hand site f the following statements are equivalent:

(1) the trivial solution of equation (1) is uniformly asymptotically stable;
(2) the trivial motion of the cocycle $\langle \mathbb{R}^n, \varphi, (H(f), \mathbb{R}, \sigma) \rangle$ generated by (1) [4, Ch.I] is uniformly asymptotically stable.

Theorem 5. *Assume that the function f is r homogeneous of degree zero and Lagrange stable.*

Then the following statements are equivalent:

(1) *the trivial solution of equation (1) is uniformly asymptotically stable;*
(2) *the trivial solution of equation (1) is globally uniformly asymptotically stable;*
(3) *there exit positive numbers $\mathcal{N}$ and ν such that*

$$\rho(\varphi(t, u, g)) \leq \mathcal{N} e^{-\nu t} \rho(u) \tag{4}$$

for any $u \in \mathbb{R}^n$, $g \in H(f)$ and $t \geq 0$, where $\rho(u) = |u|_{r,p}$.

Remark 6. 1. If the function f is τ-periodic, then the equivalence of the conditions (ii) and (iii) was established in the work [8].

2. If the function f is homogeneous of degree zero (in the classical sense, i.e., $f(t, \varepsilon x) = \varepsilon f(t, x)$ for any $\varepsilon > 0$ and $(t, x) \in \mathbb{R} \times \mathbb{R}^n$), then the equivalence of the uniform asymptotically

stability and exponential stability was established in the work [5, Ch.I] (for finite-dimensional case) and in the work [3] (for infinite-dimensional case).

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Hyper-holomorphically projective mappings of hyper-Kähler manifolds

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Definition 1. [2, 3] *A hyper-Kähler manifold* is a Riemannian manifold (M^n, g, F, G, H) with three covariant constant orthogonal automorphisms F, G, H of the tangent bundle which satisfy the quaternionic identities

$$F^2 = G^2 = G^2 = FGH = -I.$$

The symbol I denotes the identity tensor of type $(1, 1)$ in the manifold. In terms of a local coordinate system we might write:

$$F^h_\alpha F^\alpha_i = -\delta^h_i, \qquad G^h_\alpha G^\alpha_i = -\delta^h_i, \qquad H^h_\alpha H^\alpha_i = -\delta^h_i, \tag{1}$$

$$G^h_\alpha H^\alpha_i = -H^h_\alpha G^\alpha_i = F^h_i, \qquad H^h_\alpha F^\alpha_i = -F^h_\alpha H^\alpha_i = G^h_i, \qquad F^h_\alpha G^\alpha_i = -G^h_\alpha F^\alpha_i = H^h_i, \tag{2}$$

$$\nabla_j F^h_i = 0, \qquad \nabla_j G^h_i = 0, \qquad \nabla_j H^h_i = 0, \tag{3}$$

Obviously, the covariant derivative in (3) is compatible with the Riemannian metric g. In this case the metric is referred to as a hyper-Kähler one.

In a hyper-Kähler manifold, let us consider a curve $x(t)$ satisfying differential equations:

$$\frac{d^2x^h}{dt^2} + \Gamma^h_{ij}\frac{dx^i}{dt}\frac{dx^i}{dt} = \alpha(t)\frac{dx^h}{dt} + \beta(t)F^h_i\frac{dx^i}{dt} + \gamma(t)G^h_i\frac{dx^i}{dt} + \delta(t)H^h_i\frac{dx^i}{dt},$$

where $\alpha(t)$, $\beta(t)$, $\gamma(t)$ and $\delta(t)$ are certain functions of the parameter t, the symbol Γ^h_{ij} denotes connection compatible with the Riemannian metric g. We call such a curve a *hyper-holomorphically planar curve (HHP-curve).* The HHP-curves are a generalization of holomorphically planar curves [1].

Suppose two hyper-Kähler manifolds (M^n, g, F, G, H) and $(\overline{M}^n, \overline{g}, F, G, H)$ are given and the defined triple of the affinors F, G, H is the same in both manifolds.

A mapping $\pi : (M^n, g, F, G, H) \to (\overline{M}^n, \overline{g}, F, G, H)$ is an *hyper-holomorphically projective mapping (HHP-mapping)* if any HHP-curve of (M^n, g, F, G, H) is mapped under π onto an HHP-curve in $(\overline{M}^n, \overline{g}, F, G, H)$.

Theorem 2. *If two hyper-Kähler manifolds (M^n, g, F, G, H) and $(\overline{M}^n, \overline{g}, F, G, H)$ are in hyper-holomorphically projective correspondence, then their Levi-Civita connections related to each other as*

$$\overline{\Gamma}^h_{ij} = \Gamma^h_{ij} + \psi_{(i}\delta^h_{j)} - \psi_\alpha F^\alpha_{(i}F^h_{j)} - \psi_\alpha G^\alpha_{(i}G^h_{j)} - \psi_\alpha H^\alpha_{(i}H^h_{j)},$$

where ψ_i is some gradient vector.

Theorem 3. *Let a hyper-Kähler manifold (M^n, g, F, G, H) admit HHP-mappings. Then the object*

$$\overline{T}^h_{ij} = \Gamma^h_{ij} - \frac{1}{n+4}\left(\Gamma^\alpha_{\alpha(i}\delta^h_{j)} - \Gamma^\alpha_{\alpha\beta}F^\beta_{(i}F^h_{j)} - \Gamma^\alpha_{\alpha\beta}G^\beta_{(i}G^h_{j)} - \Gamma^\alpha_{\alpha\beta}H^\beta_{(i}H^h_{j)}\right)$$

is invariant under any HHP-mapping.

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On a problem of Fejes Toth

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Let P be any convex n-gon in the plane with sides $A_j, j = 1, \ldots, n$ of lengths a_j. Denote by b_j the length of the longest chord parallel to the side A_j. Fejes Tóth conjectured that $\sum_{j=1}^n \frac{a_j}{b_j} \geq 3$, with equality only for a snub triangle obtained by cutting off three congruent triangles from the corners of a triangle. This question appears as B7 in the *Unsolved Problems in Geometry* by H. T. Croft, K. J. Falconer and R. K. Guy. We will present F. Nazarov's proof of Tóth's inequality and discuss its higher-dimensional analogues.

Gottlieb groups of some Moore spaces

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In this work, we present some computations of Gottlieb groups of Moore spaces $M(A, n)$ for some classes of finitely generated abelian groups A.

Given $m \geq 1$, recall that the m-th *Gottlieb group* $G_m(X)$ of a space X has been defined in [4, 5] as the subgroup of the homotopy group $\pi_m(X)$ consisting of all elements which can be represented by a map $f\colon \mathbb{S}^m \to X$ such that $f \vee \iota_X\colon \mathbb{S}^m \vee X \to X$ extends (up to homotopy) to a map $F\colon \mathbb{S}^m \times X \to X$. Notice that $\alpha \in G_m(\Sigma X)$ if and only if the generalized Whitehead product $[\alpha, \iota_{\Sigma X}] = 0$ (see [1, Proposition 5.1]).

First, we recall from [5, Theorems 5.2 and 5.4]:

Theorem 1. *Let A be a finitely generated abelian group and $n \geqslant 3$. Then,*

$$G_n(M(A,n)) = \begin{cases} 0, & \text{if } n \text{ is even,} \\ 0, & \text{if } n \text{ is odd and } \mathrm{rk}(A) \neq 1, \\ 2\mathbb{Z} \subseteq \mathbb{Z} = \pi_n(\mathbb{S}^n), & \text{if } n \neq 1,3,7 \text{ is odd and } A = \mathbb{Z}, \\ \mathbb{Z} = \pi_n(\mathbb{S}^n), & \text{if } n = 1,3,7 \text{ and } A = \mathbb{Z} \end{cases}$$

We point out that the result above has been stated also in [2] for $n \geqslant 3$. In addition, [2, Corollary 4.4] claims that if n is odd, then $G_n(M(\mathbb{Z} \oplus T, n))$ is infinite cyclic, where T is a finite abelian group.

As stated in [2, Remark 4.5], it would be interesting to compute other Gottlieb groups for some Moore spaces, such as $G_{n+1}(M(A, n))$. We will do this for a finitely generated abelian group A which its torsion subgroup has order 2 (mod 4). We notice that on [3, Chapter 3] there are some results on $G_{n+1}(M(A, n))$ only for A having torsion subgroup with odd order.

Our main result is:

Theorem 2. *Let A be a finite abelian group with order $|A| \equiv 2 \pmod 4$. Then $G_{n+1}(M(\mathbb{Z} \oplus A, n)) = 0$, for $n \geq 3$, and $G_{n+2}(M(\mathbb{Z} \oplus A, n)) = 0$, for $n \geq 4$.*

Furthermore, investigations of $G_{n+k}(M(\mathbb{Z} \oplus A, n))$ for $k = 3, 4, 5$ and A as above, is planned as well.

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Inner semi-continuity of medial axes and conflict sets

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A central notion in pattern recognition is that of the *medial axis* M_X of a closed, nonempty, proper subset $X \subset \mathbb{R}^n$. Namely, M_X consists of all those points $a \in \mathbb{R}^n$ for which there is more than one closest point (with respect to the Euclidean distance $d(a, X)$) in X:

$$M_X := \{a \in \mathbb{R}^n \mid \#m(a) > 1\} \text{ where } m(a) := \{x \in \mathbb{R}^n \mid ||a - x|| = d(a, X)\}.$$

The definition goes back to H. Blum (cf. [3]) who gave it for $X = \partial D$ where $D \subset \mathbb{R}^n$ is a bounded domain. Then, knowing the 'skeleton' $M_X \cap D$ and $d(\cdot, X)|_{M_X}$ ('compressed data') one can reconstruct the 'shape' D.

The medial axis has long been known for being highly unstable (cf. e.g. [4]): the smallest deformation of X may lead to an important change in M_X (think of X as a circle in the plane — M_X is its centre, while the same circle but now with the smallest $\mathscr{C}^\infty$ protuberance yields a medial axis that is a segment). However, this point of view has a flaw — it sees the modification as through a blackbox, there is an initial state and a final one with nothing in between.

Our aim is to provide the right setting for considering the deformation of X which is the *(Painlevé)-Kuratowski convergence of closed sets* and to show in this case the inner-semicontinuity of the medial axis. The most general result we have, and one that turns out ot be optimal already in $\mathbb{R}^n$, can be stated as follows:

Theorem 1. *Let $\mathcal{M}$ be a connected complete Riemannian manifold and Π a T_1 topological space of parameters with a distinguished non-isolated point 0 having a countable basis of neighbourhoods. We write $\Omega_{X,p}$ for the set of geodesics of minimal length connecting a point in $m(p)$ with p and $\gamma_{X,p}$ for such a geodesic originating at p. Assume that $X \subset \Pi \times \mathcal{M}$ has closed t-sections and we have the Kuratowski convergence $X_t \xrightarrow{K} X_0$. Then for $M = \{(t, x) \in$*

$\Pi \times \mathcal{M} \mid \exists \gamma_{X_t,p}, \tilde{\gamma}_{X_t,p} \in \Omega_{X_t,p} : \gamma_{X_t,p} \neq \tilde{\gamma}_{X_t,p}\}$, *we have*

$$\liminf_{\pi(M)\ni t\to 0} M_t \supset M_0$$

where the lower limit is understood in the Kuratowski sense:

$$x \in \liminf_{\pi(M)\ni t\to 0} M_t \Leftrightarrow \forall \pi(M) \setminus \{0\} \ni t_\nu \to t_0, \exists M_{t_\nu} \ni x_\nu \to x.$$

We will show how this applies in singularity theory in $\mathbb{R}^n$ giving a criterion for M_X to reach certain singularities of X when X is definable in some o-minimal structure (e.g. semi-algebraic), cf. [2].

Finally, we will discuss a counterpart of this theorem in the case of *conflict sets* of finite families of closed, pairwise disjoint sets, instead of the medial axis, cf. [1]. The conflict set of two sets is their set of equidistant points. In case of more than two sets it can be seen as the set of points at which the distance wavefronts emanating from the sets meet.

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The diameter-width-ratio for complete and pseudo-complete sets

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Any set $A \subset \mathbb{R}^n$ fulfilling $A = t - A$ for some $t \in \mathbb{R}^n$ is called symmetric and 0-symmetric if $t = 0$. We denote the family of all (convex) bodies (full-dimensional compact convex sets) by $\mathcal{K}^n$ and the family of 0-symmetric bodies by $\mathcal{K}^n_0$. For any $K \in \mathcal{K}^n$ the gauge function $\|\cdot\|_K : \mathbb{R}^n \to \mathbb{R}$ is defined as

$$\|x\|_K = \inf\{\rho > 0 : x \in \rho K\}.$$

In case $K \in \mathcal{K}^n_0$ we see that $\|\cdot\|_K$ defines a norm. However, even for a non-symmetric unit ball K, one may approximate the gauge function by the norms induced from symmetrizations of K

$$\|x\|_{\mathrm{conv}(K\cup(-K))} \leq \|x\|_K \leq \|x\|_{K\cap(-K)}.$$

It is natural to request that $K \cap (-K) = K = \mathrm{conv}(K \cup (-K))$ if K is symmetric, which is true if and only if 0 is the center of symmetry of K. This motivates the definition of a meaningful center for general K. We introduce one of the most common asymmetry measures, which is best suited to our purposes, and choose the center matching it.

The *Minkowski asymmetry* of K (denoted by $s(K)$) is defined as

$$s(K) := \inf\{\rho > 0 : K - c \subset \rho(c - K), \quad c \in \mathbb{R}^n\},$$

and a *Minkowski center* of K is any $c \in \mathbb{R}^n$ such that $K - c \subset s(K)(c - K)$. Moreover, if 0 is a Minkowski center, we say K is *Minkowski centered.* It is well-known that $s(K) \in [1, n]$ for all $K \in \mathcal{K}^n$, with $s(K) = 1$ if and only if K is symmetric and $s(K) = n$ if and only if K is a simplex.

Notice that there always exists some $x \in \mathbb{R}^n$ such that $\alpha(K)\|x\|_{K\cap(-K)} = \|x\|_{\mathrm{conv}(K\cup(-K))}$, which means that we have equality in the complete chain in the equality above for that x if $\alpha(K) = 1$. We investigate the region of all possible values for the parameter $\alpha(K)$ for Minkowski centered $K \in \mathcal{K}^2$ in dependence of the asymmetry of K.

We show that $\alpha(K) \geq \frac{2}{s(K)+1}$ for all Minkowski centered K, and that in the planar case $\alpha(K) = 1$ implies $s(K) \leq \varphi$, where $\varphi = \frac{1+\sqrt{5}}{2} \approx 1.61$ denotes the golden ratio.

We give a complete description of the possible α-values of K in the planar case in dependence of its Minkowski asymmetry. Moreover, we derive the (unique) family of convex bodies that fulfill the upper bound of $\alpha(K)$.

K is called complete (w.r.t. C), if any proper superset of it has a greater diameter than K.

We also present an application on the diagram of the α-values of K for the diameter-width ratio for complete and pseudo-complete sets. We extend the results on the bounds for $\alpha(K)$ and describe the region of all possible values for this parameter for Minkowski centered convex compact set K in dependence of the asymmetry of K.

Theorem 1. *Let K be Minkowski centered. Then*

$$\frac{2}{s(K)+1} \leq \alpha(K) \leq \min\left\{1, \frac{s(K)}{s(K)^2 - 1}\right\}.$$

Moreover, for every pair (α, s), such that $\frac{2}{s+1} \leq \alpha \leq \min\left\{1, \frac{s}{s^2-1}\right\}$, there exists a Minkowski centered K, such that $s(K) = s$ and $\alpha(K) = \alpha$.

Consider $K \in \mathcal{K}^n$ and $C \in \mathcal{K}^n_0$. For $s \in \mathbb{R}^n \setminus \{0\}$ the s-breadth of K w.r.t. C is the distance between the two parallel supporting hyperplanes of K with normal vector s, i.e.,

$$b_s(K, C) := \frac{\max_{x,y\in K} s^T(x - y)}{\max_{x\in C} s^T x}.$$

The minimal s-breadth

$$w(K, C) := \min_{s\in mathbbR^n\setminus\{0\}} b_s(K, C)$$

and the maximal s-breadth

$$D(K, C) := \max_{s\in\mathbb{R}^n\setminus\{0\}} b_s(K, C)$$

are called width and diameter of K w.r.t. C, respectively.

We present a quantitative result on the diameter-width ratio for for complete sets.

Theorem 2. *Let K, C be convex compact sets and C be 0-symmetric be such that K is complete w.r.t. C. Then*

$$\frac{D(K, C)}{w(K, C)} \leq \frac{s(K) + 1}{2}.$$

Moreover, for $n > 2$ even and for any $s \in [1, n - 1]$ there exists $K, C \in \mathcal{K}^n$ such that K is complete w.r.t. C with $s(K) = s$, such that $\frac{D(K,C)}{w(K,C)} = \frac{s+1}{2}$, while for $n > 2$ odd and any $s \in [1, n]$ there exists $K \in \mathcal{K}^n$ which is complete w.r.t. C with $s(K) = s$, such that $\frac{D(K,C)}{w(K,C)} = \frac{s+1}{2}$.

On the possibility of joining two pairs of points in convex domains using paths

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Recall, that a set C is *convex* if any pair of points $x, y \in C$ may be joined by some segment which belongs to C, as well. We define the Euclidean distance between sets and the Euclidean diameter by the formulae

$$d(A,B)=\inf_{x\in A, y\in B}|x-y|\,,\quad d(A)=\sup_{x,y\in A}|x-y|\,.$$

Sometimes we also write $\operatorname{dist}(A,B)$ instead $d(A,B)$ and $\operatorname{diam} E$ instead $d(E)$, as well. As usually, we set

$$B(x_0,r)=\{x\in\mathbb{R}^n: |x-x_0|<r\}\,,$$
$$S(x_0,r)=\{x\in\mathbb{R}^n: |x-x_0|=r\}\,.$$

We emphasize that, the results established here have already been obtained in particular case, when a domain is the unit ball [1]. Concerning some applications of modulus inequalities in the mapping theory, see [2], cf. [3]–[4].

Theorem 1. *Let D' be a bounded convex domain in $\mathbb{R}^n$, $n\geqslant 2$, and let $E:=B(y_*,\delta_*/2)$ be a ball centered at the point $y_*\in D'$, where $\delta_*:=d(y_*,\partial D')$. Let $z_0\in\partial D'$. Then for any points $A,B\in B(z_0,\delta_*/8)\cap D'$ there are points $C,D\in\overline{B(y_*,\delta_*/2)}$, for which the segments $[A,C]$ and $[B,D]$ are such that*

$$\operatorname{dist}([A,C],[B,D])\geqslant C_0\cdot|A-B|\,,\tag{1}$$

where $C_0>0$ is some constant depending only on δ_* and $d(D')$.

Recall that, a Borel function $\rho:\mathbb{R}^n\to[0,\infty]$ is called *an admissible* for a family Γ of paths γ in $\mathbb{R}^n$, if the relation

$$\int\limits_{\gamma}\rho(x)\,|dx|\geqslant 1\tag{2}$$

holds for any locally rectifiable path $\gamma\in\Gamma$. *A modulus* of Γ is defined as follows:

$$M(\Gamma)=\inf_{\rho\in\,\mathrm{adm}\,\Gamma}\int\limits_{\mathbb{R}^n}\rho^n(x)\,dm(x)\,.\tag{3}$$

The following statements hold.

Corollary 2. *Let, under conditions of Theorem 1, Γ denotes the family of all paths joining the segments $[A,C]$ and $[B,D]$ in D'. Then*

$$M(\Gamma)\leqslant\frac{m(D')}{C_0^n}\cdot\frac{1}{|A-B|^n}\,,\tag{4}$$

where M is the modulus of families of paths defined in (3), $m(D')$ denotes the Lebesgue measure of D', and C_0 is a constant in (1).

Corollary 3. *Let, under conditions of Theorem 1, Γ denotes the family of all paths joining the segments $[A,C]$ and $[B,D]$ in D'. Then*

$$M(\Gamma) \geqslant \widetilde{c_n} \cdot \log \left(1 + \frac{3\delta_*}{8|A-B|}\right)\,, \tag{5}$$

where M is the modulus of families of paths defined in (3), $\widetilde{c_n} > 0$ is some constant depending only on n and D'.

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Backström curves

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Recall some definitions.

Definition 1. (1) A *non-commutative curve* is a pair $(X, \mathcal{A})$, where X is an algeraic curve over a field $\Bbbk$ and $\mathcal{A}$ is a sheaf of $\mathcal{O}_X$-algebras coherent as a sheaf of $\mathcal{O}_X$-modules.

(2) A non-commutative curve $(X, \mathcal{H})$ is called *hereditary* if for every point $x \in X$ the localization $\mathcal{H}_x$ is hereditary (equivalently, $\mathrm{gl.dim}\,\mathcal{H} = 1$).

(3) A non-commutative curve $(X, \mathcal{A})$ is called *Backström* if there is a hereditary non-commutative curve $(X, \mathcal{H})$ such that $\mathcal{H} \supset \mathcal{A}$ and $\mathrm{rad}\,\mathcal{H}_x = \mathrm{rad}\,\mathcal{A}_x$ for all points $x \in X$.

(4) The *Auslander envelope* of a Backström non-commutative curve $(X, \mathcal{A})$ is defined as the non-commutative curve $(X, \tilde{\mathcal{A}})$, where $\tilde{\mathcal{A}} = \mathcal{E}nd_{\mathcal{A}}(\mathcal{A} \oplus \mathcal{H})$.

For instance, every (usual) algebraic curve such that all its singularities are simple nodes is a Backström curve, as well as the union of the coordinate axes in the affine space of any dimension.

We study the structure of Backström curves and their Auslander envelopes and prove the following results.

Theorem 2. *Let $(X, \mathcal{A})$ be a Backström non-commutative curve, $(X, \tilde{\mathcal{A}})$ be its Auslander envelope.*

(1) $\mathrm{gl.dim}\,\tilde{\mathcal{A}} \leq 2$.

(2) $\mathrm{der.dim}\,\mathcal{A} \leq 2$, *where* der.dim $\mathcal{A}$ *denotes the* derived dimension *of $\mathcal{A}$, that is the Rouquier dimension* [2] *of the perfect derived category* $\mathcal{D}^{perf}(\mathrm{Coh}\,\mathcal{A})$.

Local versions of these results are proved in [1].

We also study the action of finite groups on Backström curves and prove the following theorem.

Theorem 3. *Let a finite group of order n acts on a Backström curve $(X,\mathcal{A})$ and* $\operatorname{char}\Bbbk \nmid n$. *Then the crossed product $(X,\mathcal{A}*G)$ is also a Backström curve and its Auslander envelope is $(X,\tilde{\mathcal{A}}*G)$.*

Some examples will also be presented.

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On geodesic lines of Riemannian metric for Navier-Stokes equations

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Theorem 1. *The 14D Riemann metric in local coordinates*

$$\vec{x} = (x, y, z, t, \eta, \rho, m, u, v, w, p, \xi, \chi, n)$$

$$\begin{aligned} ds^2 &= 2\,dxdu + 2\,dydv + 2\,dzdw + (-W(\vec{x},t)w - V(\vec{x},t)v - U(\vec{x},t)u)\,dt^2 + \\ &+ \left(-U(\vec{x},t)p - u\,(U(\vec{x},t))^2 - uP(\vec{x},t) + w\mu\,\frac{\partial}{\partial z}U(\vec{x},t) - wU(\vec{x},t)W(\vec{x},t)\right) d\eta^2 + \\ &+ \left(v\mu\,\frac{\partial}{\partial y}U(\vec{x},t) - vU(\vec{x},t)V(\vec{x},t) + u\mu\,\frac{\partial}{\partial x}U(\vec{x},t)\right) d\eta^2 + 2\,d\eta d\xi + 2\,d\rho d\chi + 2\,dmdn + \\ &+ \left(-V(\vec{x},t)p - vP(\vec{x},t) - v\,(V(\vec{x},t))^2 - V(\vec{x},t)W(\vec{x},t)w + v\mu\,\frac{\partial}{\partial y}V(\vec{x},t) - uU(\vec{x},t)V(\vec{x},t)\right) d\rho^2 + \\ &+ \left(u\mu\,\frac{\partial}{\partial x}V(\vec{x},t)\right) d\rho^2 + \left(-uU(\vec{x},t)W(\vec{x},t) - w\,(W(\vec{x},t))^2 - wP(\vec{x},t) + w\mu\,\frac{\partial}{\partial z}W(\vec{x},t)\right) dm^2 + \\ &+ \left(v\mu\,\frac{\partial}{\partial y}W(\vec{x},t) - vV(\vec{x},t)W(\vec{x},t) + u\mu\,\frac{\partial}{\partial x}W(\vec{x},t) - W(\vec{x},t)p\right) dm^2 \end{aligned} \tag{1}$$

is the Ricci-flat,

$$R_{44} = U_x + V_y + W_z = 0,\ R_{55} = 0,\ R_{66} = 0,\ R_{77} = 0$$

on solutions of Navier-Stokes system of equations

$$\frac{\partial}{\partial t}\vec{Q}(\vec{x},t) + (\vec{Q}(\vec{x},t)\cdot\vec{\nabla})\vec{Q}(\vec{x},t) - \mu\,\Delta\vec{Q}(\vec{x},t) + \vec{\nabla}P(\vec{x},t) = 0, \quad \vec{\nabla}\cdot\vec{Q}(\vec{x},t) = 0, \tag{2}$$

where $\vec{Q}(\vec{x},t) = [U(\vec{x},t), V(\vec{x},t), W(\vec{x},t)]$ *are the components of velocity and* $P(\vec{x},t)$ *is pressure of liquid.(see e.g. [1-2])*

To obtain the metric (1) presentation the NS-system of equations in the form of laws conservations

$$U_t + (U^2 - \mu U_x + P)_x + (UV - \mu U_y)_y + (UW - \mu U_z)_z = 0$$
$$V_t + (V^2 - \mu V_y + P)_y + (UV - \mu V_x)_x + (VW - \mu V_z)_z = 0,$$
$$W_t + (W^2 - \mu W_z + P)_z + (UW - \mu W_x)_x + (VW - \mu W_y)_y = 0,$$
$$(U_x + V_y + W_z) = 0,$$

is used.

The metric (1) belongs to the class of the Riemann spaces with vanishing scalar Invariants. Their geodesics with respect to the coordinates $\eta, \rho, m, \xi, \chi, n$ has form of equations direct lines

$$\ddot{\eta} = 0,\ \ddot{\rho} = 0,\ \ddot{m} = 0,\ \ddot{\xi} = 0,\ \ddot{\chi} = 0,\ \ddot{n} = 0,$$

and in this sense to them the partially-projective spaces of V.Kagan corresponds.

For the coordinates $[x, y, z, t]$ the equations of geodesics of metric (1) are

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}x(s) = 1/2\ (\dot{m}(s))^2 U(x,y,z,t)\, W(x,y,z,t) - 1/2\ (\dot{m}(s))^2 \mu \frac{\partial}{\partial x}W(x,y,z,t) +$$
$$+ 1/2\ (\dot{\eta}(s))^2 (U(x,y,z,t))^2 + 1/2\ (\dot{\rho})^2 U(x,y,z,t)\, V(x,y,z,t) -$$
$$- 1/2\ (\dot{\rho})^2 \mu \frac{\partial}{\partial x}V(x,y,z,t) - -1/2\ (\dot{\eta})^2 \mu \frac{\partial}{\partial x}U(x,y,z,t) +$$
$$+ 1/2\, U(x,y,z,t) \left(\frac{\mathrm{d}}{\mathrm{d}s}t(s)\right)^2 + 1/2\ (\dot{\eta}(s))^2 P(x,y,z,t),$$

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}t(s) = 1/2\, W(x,y,z,t) \left(\frac{\mathrm{d}}{\mathrm{d}s}m(s)\right)^2 + 1/2\, U(x,y,z,t) \left(\frac{\mathrm{d}}{\mathrm{d}s}\eta(s)\right)^2 +$$
$$+ 1/2\, V(x,y,z,t) \left(\frac{\mathrm{d}}{\mathrm{d}s}\rho(s)\right)^2,$$

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}y(s) = \cdots,$$
$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}z(s) = \cdots.$$

The equations of geodesics for dual coordinates $[u, v, w, p]$ form the linear system of the second order equations

$$\frac{d^2}{ds^2}u(s) = A_1 u(s) + B_1 v(s) + C_1 w(s) + E_1 p(s),$$
$$\frac{d^2}{ds^2}v(s) = A_2 u(s) + B_2 v(s) + C_2 w(s) + E_2 p(s),$$
$$\frac{d^2}{ds^2}w(s) = A_3 w(s) + B_3 v(s) + C_3 w(s) + E_3 p(s),$$
$$\frac{d^2}{ds^2}p(s) = A_4 u(s) + B_4 v(s) + C_4 w(s) + E_4 p(s),$$

with the coefficients depending on the solutions $U(x,y,z,t), V(x,y,z,t), W(x,y,z,t), P(x,y,z,t$ of the system (2).

On the base of solutions of equations for the Killing vectors of the metric

$$K_{i,j}+K_{j,i}-2\Gamma^k_{ij}K_k=0, \quad or \quad K^k g_{ij,k}+g_{ik}K^k,j+g_{jk}K^k,i=0, \tag{4}$$

a new examples of reductions and solutions of the system (2) are constructed.

Properties of the Lie derivative for the connection coefficients of the metric (1) and the vector field of the form $u^i=g^i_k v^k$

$$u^i_{j,k}+u^n\Gamma^i_{jk},_n+u^n_{,j}\Gamma^i_{nk}+u^n_{,k}\Gamma^i_{jn}-u^n_{,n}\Gamma^i_{jk}=0,$$

where Γ^i_{jk}-are the coefficients of connection of the metric (1) with the aim of constructing new examples of solutions to the system (2) are discussed.

Another possibility for studying the properties of the NS system by the geometric method is the use of differential Beltrami parameters of the metric (1) $\Delta_2(f)=g^{ij}\frac{\partial^2 f}{\partial x_i,x_j}-\Gamma^k_{ij}\frac{\partial f}{\partial x_k}$. As example, in particular case $f=\psi(x,y,z,t,0,0,0,u,v,w,p,0,0,0)$, from solutions of the linear equation with variable coefficients $\Delta_2(f)=0$ the relation

$$(U(\vec{x},t)-W(\vec{x},t))P(\vec{x},t)/\mu=U(\vec{x},t)\frac{\partial}{\partial z}U(\vec{x},t)-W(\vec{x},t)\frac{\partial}{\partial x}V(\vec{x},t)-$$
$$-W(\vec{x},t)\frac{\partial}{\partial x}U(\vec{x},t)-W(\vec{x},t)\frac{\partial}{\partial x}W(\vec{x},t)+\frac{\partial}{\partial z}V(\vec{x},t)U(\vec{(}x,t)+\frac{\partial}{\partial z}W(\vec{x},t)U(\vec{x},t),$$

between velocity and pressure can be derived and that can be applied to the studying properties of solutions of the system (2).

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On controllability problems for the heat equation in a half-plane in the case of a pointwise control in the Dirichlet boundary condition

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Consider the following control system in a half-plane

$$w_t = \Delta w, \qquad x_1 > 0,\ x_2 \in \mathbb{R},\ t \in (0,T), \tag{1}$$

$$w\big(0,(\cdot)_{[2]},t\big) = \delta_{[2]} u(t), \qquad x_2 \in \mathbb{R},\ t \in (0,T), \tag{2}$$

$$w\big((\cdot)_{[1]},(\cdot)_{[2]},0\big) = w^0, \qquad x_1 > 0,\ x_2 \in \mathbb{R}, \tag{3}$$

where $T > 0$, $u \in L^\infty(0,T)$ is a control, $\delta_{[m]}$ is the Dirac distribution with respect to x_m, $m = 1,2$, $\Delta = (\partial/\partial x_1)^2 + (\partial/\partial x_2)^2$. The subscripts [1] and [2] associate with the variable numbers, e,g., $(\cdot)_{[1]}$ and $(\cdot)_{[2]}$ correspond to x_1 and x_2, respectively.

Let $\mathbb{R}_+ = (0,+\infty)$. Consider the following spaces of Sobolev type

$$H^s_{⓪} = \Bigg\{\varphi \in L^2(\mathbb{R}_+\times\mathbb{R}) \,\Bigg|\, \Bigg(\forall \alpha = (\alpha_1,\alpha_2) \in \mathbb{N}_0^2\ \Bigg(\alpha_1+\alpha_2 \le s \Rightarrow \frac{\partial^{\alpha_1+\alpha_2}\varphi}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}} \in L^2(\mathbb{R}_+\times\mathbb{R})\Bigg)\Bigg) \wedge\Bigg(\forall k = \overline{0,s-1}\ \frac{\partial^k \varphi(0^+,(\cdot)_{[2]})}{\partial x_1^k} = 0\Bigg)\Bigg\}, \quad s = \overline{0,3},$$

with the norm

$$\|\varphi\|^s_{⓪} = \left(\sum_{\alpha_1+\alpha_2\le s}\left(\left\|\frac{\partial^{\alpha_1+\alpha_2}\varphi}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}}\right\|_{L^2(\mathbb{R}_+\times\mathbb{R})}\right)^2\right)^{1/2}, \quad \varphi \in H^s_{⓪}, \quad s = \overline{0,3},$$

and $H^{-s}_{⓪} = \big(H^s_{⓪}\big)^*$ with the strong norm $\|\cdot\|^{-s}_{⓪}$ of the adjoint space. We have $H^0_{⓪} = L^2(\mathbb{R}_+\times\mathbb{R})$.

We consider control system (1)–(3) in $H^{-l}_{⓪}$, $l = \overline{1,3}$, i.e. $\left(\frac{d}{dt}\right)^s w : [0,T] \to H^{-1-2s}_{⓪}$, $s = 0,1$, $w^0 \in H^{-1}_{⓪}$. We treat equality (2) as the value of the distribution w at $x_1 = 0$ (see the definition of a distribution's value at a point [1, Chap. 1] and the definition of a distribution's value at a line [2]).

Definition 1. A state $w^0 \in H^{-1}_{⓪}$ is said to be controllable to a target state $w^T \in H^{-1}_{⓪}$ in a given time $T > 0$ if there exists a control $u \in L^\infty(0,T)$ such that there exists a unique solution w to system (1)–(3) and $w\big((\cdot)_{[1]},(\cdot)_{[2]},T\big) = w^T$.

Definition 2. A state $w^0 \in H_{\circledcirc}^{-1}$ is said to be approximately controllable to a target state $w^T \in H_{\circledcirc}^{-1}$ in a given time $T > 0$ if for each $\varepsilon > 0$, there exists $u_\varepsilon \in L^\infty(0,T)$ such that there exists a unique solution w_ε to system (1)–(3) with $u = u_\varepsilon$ and $\left\| w_\varepsilon\big((\cdot)_{[1]}, (\cdot)_{[2]}, T\big) - w^T \right\|_{\circledcirc}^{-1} < \varepsilon$.

The main goal of the paper is to study whether the state w^0 is controllable (approximately controllable) to a target state w^T in the time T.

Note that controllability problems for the heat equation in domains bounded with respect to spatial variables were investigated rather completely in a number of papers. However, these problems for the heat equation in domains unbounded with respect to spatial variables have not been fully studied.

For control system (1)–(3), the following assertions are obtained in a given time $T > 0$ under the control bounded by a given constant ($|u(t)| \le U$, $t \in [0,T]$): a necessary condition for controllability from the origin; necessary and sufficient conditions for controllability; sufficient conditions for approximate controllability in terms of Markov power moment problem constructed according to the control problem data.

Using the generalised Laguerre polynomials, we also construct orthogonal bases in special spaces of Sobolev type. With the aid of the constructed bases, we obtain necessary and sufficient conditions for approximate controllability in a given time for system (1)–(3) in the case of L^∞-control. The results are illustrated by an example:

Example 3. Let $T = 1/2$,

$$w^0(x) = \frac{x_1}{T^2} e^{-\frac{|x|^2}{4T}}, \quad w^T(x) = \frac{x_1}{8T^2} e^{-\frac{|x|^2}{8T}}, \quad x_1 > 0,\ x_2 \in \mathbb{R}.$$

Verifying the obtained necessary and sufficient conditions for approximate controllability in a given time for system (1)–(3), we conclude that the state w^0 is approximately controllable to the state w^T in the time $T = 1/2$. Using the algorithm given in [3], we construct end states $w_l^N(\cdot, T) \in H_{\circledcirc}^{-1}$ and piecewise constant controls $u_{N,l}$ depending on two parameters N and l, $l = \overline{2(N+2), \infty}$, $N = \overline{1, \infty}$, such that

$$\left\| w_l^N(\cdot, T) - w^T \right\|_{\circledcirc}^{-1} \to 0, \quad \text{as } N \to \infty,\ l \to \infty.$$

All obtained results have been published in [3].

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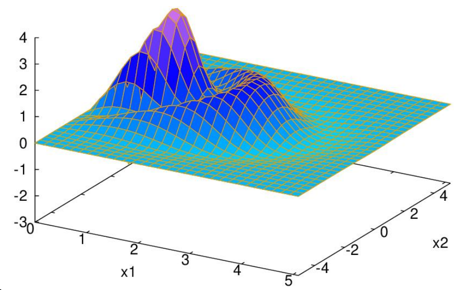
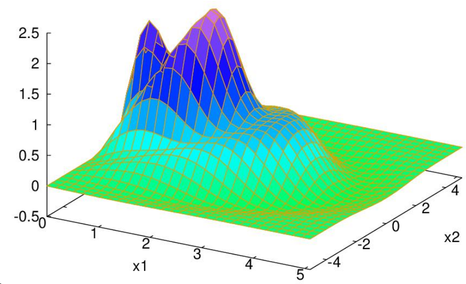


4cm

(a) t

FIGURE 3.1. $w_l^N(\cdot, T) - w^T$, $N = 3$, $l = 50$.



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(a)

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FIGURE 3.2. $w_l^N(\cdot, T) - w^T$, $N = 4$, $l = 200$.

FIGURE 3.3. The influence of the controls $u_{N,l}$ on the difference $w_l^N(\cdot, T) - w^T$.

On partial preliminary group classification of some class of $(1 + 3)$-dimensional Monge-Ampere equations. Two-dimensional Abelian Lie algebras

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Classes of Monge-Ampère equations, in the spaces of different dimensions and different types, arise in solving of many problems of the geometry, theoretical physics, optimal mass transportation, geometric optics, one-dimensional gas dynamics and etc.

At the present time, there are a lot of papers and books in which those classes have been studied by different methods.

We consider the following class of $(1+3)$-dimensional Monge-Ampère equations:

$$\det\left(u_{\mu\nu}\right) = F\left(x_0, x_1, x_2, x_3, u, u_0, u_1, u_2, u_3\right),$$

where $u = u(x)$, $x = (x_0, x_1, x_2, x_3) \in M(1,3)$, $u_{\mu\nu} \equiv \dfrac{\partial^2 u}{\partial x_\mu \partial x_\nu}$, $u_\alpha \equiv \dfrac{\partial u}{\partial x_\alpha}$, $\mu, \nu, \alpha = 0, 1, 2, 3.$

Here, $M(1,3)$ is a four-dimensional Minkowski space, F is an arbitrary real smooth function.

For the group classification of this class we have used the classical Lie-Ovsiannikov approach. At the present time, we have performed partial preliminary group classification of the class under investigation, using two-dimensional Abelian nonconjugate subalgebras of the Lie algebra of the Poincaré group $P(1,4)$.

In our report, I plan to present some of the results obtained concerning with partial preliminary group classification of the class under consideration.

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Homotopy type of stabilizers of functions with non-isolated singularities on surfaces

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Let M be a smooth compact surface, $\mathcal{D}(M)$ be a group of diffeomorphisms of M, and P be either $\mathbb{R}$ or S^1. For a smooth function $f : M \to P$ denote by $\mathcal{S}(f)$ a group of f-preserving diffeomorphisms of M, i.e.,

$$\mathcal{S}(f) = \{h \in \mathcal{D}(M) \mid f \circ h = f\},$$

and by $\mathcal{S}_{\mathrm{id}}(f)$ a connected component of $\mathcal{S}(f)$ containing id_M.

In [1] the author considered the following class of functions $\mathcal{F}(M,P)$ and described the homotopy type of $\mathcal{S}_{\mathrm{id}}(f)$ for functions from it.

Definition 1. A smooth function $f \in C^\infty(M,P)$ on M belongs to the class $\mathcal{F}(M,P)$ if the following conditions are satisfied:

(1) for each connected component V of the boundary ∂M a function $f|_V$ either takes a constant value or is a covering map,

(2) a set of critical points Σ_f of f is a disjoint union of smooth submanifolds of M and $\Sigma_f \subset \mathrm{Int}(M)$,
(3) for each connected component C of Σ_f and each critical point $p \in C$ there exist a local chart $(U, \phi : U \to \mathbb{R}^2)$ near p and a chart $(V, \psi : V \to \mathbb{R})$ near $f(p) \in P$ such that $f(U) \subset V$ and a local representation $\psi \circ f \circ \phi^{-1} : \phi(U) \to \psi(V)$ of f is
  (a) either a homogeneous polynomial $f_p : \mathbb{R}^2 \to \mathbb{R}$ of degree $\deg f_p \geq 2$ having no multiple factors,
  (b) or is given by $f_C(x, y) = \pm y^{n_C}$ for some $n_C \in \mathbb{N}_{\geq 2}$ depending of C.

Note that the class $\mathcal{F}(M, P)$ contains the class of P-valued Morse-Bott functions on M.

Theorem 2 (Theorem 1.2 [1]). *For a function $f \in \mathcal{F}(M, P)$ the group $\mathcal{S}_{\mathrm{id}}(f)$ is contractible if f has at least one saddle or M is non-oriented, otherwise $\mathcal{S}_{\mathrm{id}}(f)$ is homotopy equivalent to S^1.*

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On direct limits of Minkowski's balls, domains, and their critical lattices

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We construct direct systems of Minkowski, Davis and Chebyshev-Cohn balls and domains, direct systems of their critical lattices and calculate their direct limits. By (general) Minkowski balls we mean (two-dimensional) balls in $\mathbb{R}^2$ of the form

$$D_p: \ |x|^p + |y|^p \leq 1, \ p \geq 1. \tag{1}$$

From the proof of Minkowski's conjecture [1, 2, 3, 4, 5, 8] in notations [8, 9] we have next expressions for critical determinants and their lattices:

Theorem 1. (1) $\Delta(D_p) = \Delta_p^{(0)} = \Delta(p, \sigma_p) = \frac{1}{2}\sigma_p, \ 2 \leq p \leq p_0;$
(2) $\sigma_p = (2^p - 1)^{1/p},$
(3) $\Delta(D_p) = \Delta_p^{(1)} = \Delta(p, 1) = 4^{-\frac{1}{p}} \frac{1+\tau_p}{1-\tau_p}, \ 1 \leq p \leq 2, \ p \geq p_0,$
(4) $2(1 - \tau_p)^p = 1 + \tau_p^p, \ 0 \leq \tau_p < 1,$
where p_0 is a real number that is defined unique by conditions $\Delta(p_0, \sigma_p) = \Delta(p_0, 1), \ 2,57 < p_0 < 2,58, \ p_0 \approx 2.5725$
For their critical lattices respectively $\Lambda_p^{(0)}, \ \Lambda_p^{(1)}$ next conditions satisfy: $\Lambda_p^{(0)}$ and $\Lambda_p^{(1)}$ are two D_p-admissible lattices each of which contains three pairs of points on the boundary of D_p with the property that $(1, 0) \in \Lambda_p^{(0)}, \ (-2^{-1/p}, 2^{-1/p}) \in \Lambda_p^{(1)},$

Denote by $V(D_p)$ the volume (area) of D_p.

Proposition 2. *The volume of Minkowski ball* D_p *is equal* $4\frac{(\Gamma(1+\frac{1}{p}))^2}{\Gamma(1+\frac{2}{p})}$.

Proof. (by Minkowski). Let $x^P + y^p \leq 1, x \geq 0, y \geq 0$. Put $x^p = \xi, y^p = \eta$.

$$V(D_p) = \frac{4}{p^2} \iint \xi^{\frac{1}{p}-1} \eta^{\frac{1}{p}-1} d\xi d\eta, \tag{2}$$

where the integral extends to the area

$$\xi + \eta \leq 1,\ \xi \geq 0,\ \eta \geq 0.$$

Expression (2) can be represented in terms of Gamma functions, and we get

$$V(D_p) = 4\frac{(\Gamma(1+\frac{1}{p}))^2}{\Gamma(1+\frac{2}{p})}.$$

□

We consider balls of the form

$$D_p:\ |x|^p + |y|^p \leq 1,\ p \geq 1,$$

and call such balls with $1 < p < 2$ *Minkowski balls*. Continuing this, we consider the following classes of balls and circles.

- *Davis balls*: $|x|^p + |y|^p \leq 1$ for $p_0 > p \geq 2$;
- *Chebyshev-Cohn balls*: $|x|^p + |y|^p \leq 1$ for $p \geq p_0$;

Let D be a fixed bounded symmetric about origin convex body (*centrally symmetric convex body* for short) with volume $V(D)$.

Proposition 3. [6]. *If* D *is symmetric about the origin and convex, then* $2D$ *is convex and symmetric about the origin.*

Corollary 4. *Let* m *be integer* $m \geq 0$ *and* n *be natural greater* m. *If* $2^m D$ *centrally symmetric convex body then* $2^n D$ *is again centrally symmetrc convex body.*

Proof. Induction.

We consider the following classes of balls (see above) and domains.

- *Minkowski domains*: $2^m D_p$, integer $m \geq 1$, for $1 \leq p < 2$;
- *Davis domains*: $2^m D_p$, integer $m \geq 1$, for $p_0 > p \geq 2$;
- *Chebyshev-Cohn domains*: $2^m D_p$, integer $m \geq 1$, for $p \geq p_0$;

Proposition 5. *Let* m *be integer,* $m \geq 1$. *If* Λ *is the critical lattice of the ball* D_p *than the sublattice* Λ_{2^m} *of index* 2^m *is the critical lattice of the domain* $2^{m-1} D_p$.

The direct system of Minkowski balls and domains has the form (3), where the multiplication by 2 is the continuous mapping

$$D_p \xrightarrow{2} 2D_p \xrightarrow{2} 2^2 D_p \xrightarrow{2} \cdots \xrightarrow{2} 2^m D_p \xrightarrow{2} \cdots \tag{3}$$

The direct system of critical lattices has the form (4), where the multiplication by 2 is the homomorphism of abelian groups

$$\Lambda_p \xrightarrow{2} 2\Lambda_p \xrightarrow{2} 2^2 \Lambda_p \xrightarrow{2} \cdots \xrightarrow{2} 2^m \Lambda_p \xrightarrow{2} \cdots \tag{4}$$

In our considerations we have direct systems of Minkowski balls, Minkowski domains and direct systems of critical lattices with respective maps and homomorphisms. Let $\mathbb{Q}_2$ and $\mathbb{Z}_2$ be respectively the field of 2-adic numbers and its ring of integers. Denote the corresponding direct limits by D_p^{dirlim} and by Λ_p^{dirlim}.

Proposition 6. $D_p^{dirlim} = \varinjlim 2^m D_p \in (\mathbb{Q}_2/\mathbb{Z}_2)D_p = (\bigcup_m \frac{1}{2^m}\mathbb{Z}_2/\mathbb{Z}_2)D_p$.

Proposition 7. $\Lambda_p^{dirlim} = \varinjlim 2^m \Lambda_p \in (\mathbb{Q}_2/\mathbb{Z}_2)\Lambda_p = (\bigcup_m \frac{1}{2^m}\mathbb{Z}_2/\mathbb{Z}_2)\Lambda_p$.

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On KB(Kantorovich-Banach) spaces and KB operators

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Let E be a Banach lattice and X be a Banach space. E is said to be a KB space if a positive increasing sequence in the closed unit ball of E converges. Every KB-space has order continuous norm, but the converse is not true in general. c_0 has order continuous norm, but c_0 is not a KB-space. For $1 \leq p < \infty$, L^p-spaces are KB-spaces.

An operator $T : E \to X$ is said to be a KB operator if for every positive increasing sequence (x_n) in the closed unit ball of E , the sequence (Tx_n) converges. An operator $T : X \to X$ is called demicompact if, for every bounded sequence (x_n) in X such that $(x_n - Tx_n)$ converges to $x \in X$, there is a convergent subsequence of (x_n). An operator $T : X \to X$ is said to be a demi Dunford-Pettis if, for every sequence (x_n) in X such that (x_n) converges to zero weakly and $\|x_n - Tx_n\| \to 0$ as $n \to \infty$, we have $\|x_n\| \to 0$ as $n \to \infty$. Every Dunford-Pettis operator is demi Dunford-Pettis operator. An operator $T : E \to E$ is called a demi KB operator if, for every positive increasing sequence (x_n) in the closed unit ball of E such that $(x_n - Tx_n)$ is norm convergent to $x \in E$, there is a norm convergent subsequence of (x_n). For the identity operator $I : E \to E$, the operator $2I$ is a demi KB-operator. Every KB operator is a demi KB operator.

Definition 1. Let E be a Banach lattice. An operator $T : E \to E$ is said to be an unbounded demi KB operator if, for every positive increasing sequence (x_n) in the closed unit ball of E such that $(x_n - Tx_n)$ is unbounded norm convergent to $x \in E$, there is an unbounded norm convergent subsequence of (x_n).

Theorem 2. *Let E be a Banach lattice. Every KB operator $T : E \to E$ is unbounded demi KB operator.*

In this study, we characterize the operators on Banach lattices that under which conditions they satisfy unbounded demi KB operators.

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On polynomial and regular maps of spheres

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This talk offers some results on to the intersection of algebraic topology and algebraic geometry.

Let K be a field and $X \subseteq K^m$, $Y \subseteq K^n$ algebraic sets. Recall that a map $f = (f_1, \ldots, f_n) : X \to Y$ is called *polynomial* (resp. *regular*) if there are polynomials $F_i, G_i \in \mathbb{R}[X_1, \ldots, X_m]$ such that $f_i(x) = F_i(x)$ (resp. $f_i(x) = \frac{F_i(x)}{G_i(x)}$, $G_i(X) \neq 0$) with $i = 1, \ldots, n$ for $x \in X$.

Remark 1. If K is a algebraically closed field then the only regular maps of algebraic sets are polynomial maps.

Example 2. (1) Let $K = \mathbb{R}$ or $\mathbb{C}$, the fields of reals or complex numbers. The n-sphere

$$\mathbb{S}^n(K) = \{(x_1, \ldots, x_{n_1}) \in \mathbb{K}^{n+1};\, x_1^2 + \cdots + x_{n+1}^2 = 1\} = V(X_0^2 + \cdots + X_n^2 - 1)$$

is an algebraic set in $\mathbb{K}^{n+1}$. Write $\mathbb{S}^n(\mathbb{R}) = \mathbb{S}^n$ and notice a diffeomorphism $\mathbb{S}^n(\mathbb{C}) \approx T\mathbb{S}^n$, the tangent bundle of $\mathbb{S}^n$. Consequently, a homotopy equivalence $\mathbb{S}^n(\mathbb{C}) \simeq \mathbb{S}^n$.

(2) Let $K = \mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ with the skew $\mathbb{R}$-algebra $\mathbb{H}$ of quaternions. The Grassmannian (of r-planes in K^n), can be identified with $G_{n,r}(K) = \{A \in M_n(K);\, A^2 = A,\, \bar{A} = A^t,\, \mathrm{rk}(A) = r\}$ for the set $M_n(K)$ of all $n \times n$-matrices over K.

But, for any idempotent $n \times n$ matrix over K, its rank coincides with the trace. Therefore, $G_{n,r}(K)$ can be viewed as a real affine variety.

Let $X \subseteq \mathbb{R}^m$, $Y \subseteq \mathbb{R}^n$ be algebraic sets. Write $[X, Y]$ for the set of homotopy classes of continuous maps and $[X, Y]_{alg}$ the subset of $[X, Y]$ represented by regular maps. One of the main purposes of the talk is to estimate the size of $\pi_m(\mathbb{S}^n)_{alg} = [\mathbb{S}^m, \mathbb{S}^n]_{alg}$ in $\pi_m(\mathbb{S}^n) = [\mathbb{S}^m, \mathbb{S}^n]$.

Basing on [1], [3] and [4], we aim to show:

Theorem 3. *If $k = 0, 1, \dots, 7$ then elements of $\pi_{n+k}(\mathbb{S}^n)$ can be represented by regular maps for $n \geq 1$.*

Next, we make use of [2] to show a homeomorphism $TG_{n,r}(K) \xrightarrow{\approx} \mathrm{Idem}_{n,r}(K)$ for the tangent bundle $TG_{n,r}(K)$ of $G_{n,r}(K)$ and $\mathrm{Idem}_{n,r}(K)$, the set of all idempotent $n \times n$ matrices with rank r for $K = \mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$. Finally, we present:

Theorem 4. *If $K = \mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ then there is:*

(1) *a regular deformation retraction* $\mathrm{Idem}_{n,r}(K) \to G_{n,r}(K)$;

(2) *an injection* $\mathcal{P}_{\mathbb{C}}[V_{\mathbb{C}}, \mathrm{Idem}_{n,r}(K)] \to \mathcal{R}_{\mathbb{R}}[V, G_{n,r}(K)]$ *from the sets of homotopy classes of complex-valued polynomial to such a set of real-valued regular maps, where $V_{\mathbb{C}}$ denotes the Zariski closure in the affine space $\mathbb{C}^n$ of a subset $V \subseteq \mathbb{R}^n$.*

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On homomorphisms of bicyclic extensions of archimedean totally ordered groups

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We follow the terminology of [1, 2]. Let G^+ be the positive cone of a totally ordered group. On the set $\mathscr{B}^+(G)) = G^+ \times G^+$ we define the semigroup operation "$\cdot$" in the following way

$$(a,b)\cdot(c,d) = \begin{cases} (c\cdot b^{-1}\cdot a, d), & \text{if } \ b<c; \\ (a,d), & \text{if } \ b=c; \\ (a, b\cdot c^{-1}\cdot d), & \text{if } \ b>c, \end{cases}$$

for $a, b, c, d \in G^+$.

Theorem 1. *Let G and H be archimedean totally ordered groups. Then every o-homomorphism $\widehat{\varphi}\colon G \to H$ generates a monoid homomorphism $\widetilde{\varphi}\colon \mathscr{B}^+(G) \to \mathscr{B}^+(H)$, and every monoid homomorphism $\widetilde{\varphi}\colon \mathscr{B}^+(G) \to \mathscr{B}^+(H)$ generates an o-homomorphism $\widehat{\varphi}\colon G \to H$, which agree* *according to the formula*

$$(x,y)\widetilde{\varphi} = \big((x)\widehat{\varphi}, (y)\widehat{\varphi}\big), \qquad x,y \in G^+.$$

Theorem 2. *Let G be an archimedean totally ordered group. Then the semigroup $\mathbf{End}^o(G)$ of o-endomorphisms of G is isomorphic to the semigroup $\mathbf{End}(\mathscr{B}^+(G))$ of endomorphisms of the monoid $\mathscr{B}^+(G)$.*

We define the category $\mathfrak{TOAG}$ by

(1) $\mathbf{Ob}(\mathfrak{TOAG}) = \{G\colon G$ is an archimedean totally ordered group$\}$;
(2) $\mathbf{Mor}(\mathfrak{TOAG})$ are o-homomorphisms of archimedean totally ordered groups,

and the category $\mathfrak{BETOAG}$ in the following way

(1) $\mathbf{Ob}(\mathfrak{BETOAG})$ are bicyclic extensions $\mathscr{B}^+(G)$ of archimedean totally ordered groups $G \in \mathbf{Ob}(\mathfrak{TOAG})$;
(2) $\mathbf{Mor}(\mathfrak{BETOAG})$ are homomorphisms of monoids $\mathscr{B}^+(G) \in \mathbf{Ob}(\mathfrak{BETOAG})$.

Theorem 3. *The categories $\mathfrak{TOAG}$ and $\mathfrak{BETOAG}$ are isomorphic.*

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The Interaction of an Infinite Number of Eddy Flows

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The Boltzmann kinetic equation plays an important role in the kinetic theory of gases. In paper [2], we consider this equation for a model of hard spheres that describes particles of any gas which move translationally with a certain linear velocity, collide by the laws of classical mechanics and can not rotate. For this model, the equation has the form [1]

$$D(f) = Q(f,f), \tag{1}$$

$$D(f) \equiv \frac{\partial f}{\partial t} + \left(V, \frac{\partial f}{\partial x}\right), \tag{2}$$

$$Q(f,f) \equiv \frac{d^2}{2}\int_{R^3} dV_1 \int_{\Sigma} d\alpha |(V - V_1, \alpha)| \times \Big[f(t,x,V_1')f(t,x,V') - f(t,x,V)f(t,x,V_1)\Big], \tag{3}$$

and V, V_1, V', V_1' are the velocities of particles before and after collision, respectively, determined by the relations

$$V' = V - \alpha(V - V_1, \alpha),$$
$$V_1' = V_1 + \alpha(V - V_1, \alpha).$$

The solution to this equation will be look for in the next form

$$f(t,x,V)=\sum_{i=1}^{\infty}\varphi_i(t,x)M_i(t,x,V). \tag{4}$$

where $M_i(t,x,V)$ are the exact solutions of the equation (1)-(3)

$$D(M_i)=Q(M_i,M_i)=0$$

and the coefficient functions $\varphi_i(t,x)$ are nonnegative smooth functions on $\mathbb{R}^4$ and $\varphi_i(t,x)\not\equiv 0$.

As a value of the deviation between the parts of equation (1) we will consider a uniform-integral error of the form

$$\Delta=\Delta(\beta_i)=\sup_{(t,x)\in\mathbb{R}^4}\int_{\mathbb{R}^3}\big|D(f)-Q(f,f)\big|dV. \tag{5}$$

In the paper [2], several cases of coefficient functions $\varphi_i(t,x)$ were obtained for which the deviation (5) can be done arbitrarily small. This is possible thanks to a special selection of hydrodynamic flow parameters.

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Semi-Fredholm theory in unital C^*-algebras

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The Fredholm and semi-Fredholm theory on Hilbert and Banach spaces started by studying the integral equations introduced in the pioneering work by Fredholm in 1903 in [5]. After that, the abstract theory of Fredholm and semi-Fredholm operators on Hilbert and Banach spaces was further developed in numerous papers and books such as [1], [2] and [14]. In addition to classical semi-Fredholm theory on Hilbert and Banach spaces, several generalizations of this theory have been considered. Breuer for example started the development of Fredholm theory in von-Neumann algebras as a generalization of the classical Fredholm theory for operators on Hilbert spaces. In [3] and [4] he introduced the notion of a Fredholm operator in a von Neumann algebra and established its main properties. On the other hand, Fredholm theory on Hilbert C^*-modules as another generalization of the classical Fredholm theory on Hilbert spaces was started by Mishchenko and Fomenko. In [13] they introduced the notion of a Fredholm operator on the standard Hilbert C^*-module and proved a generalization in this setting of some of the main results from the classical Fredholm theory. In [6], [7], [8], [9] and [10] we went further in this direction and defined semi-Fredholm and semi-Weyl operators on Hilbert C^*-modules. We investigated and proved several properties of these new semi- Fredholm operators on Hilbert C^*-modules as a generalization of the results from the classical semi-Fredholm theory on Hilbert and Banach spaces. The interest for considering these generalizations comes from the theory of pseudo differential operators acting on manifolds. The classical theory can be applied in the case of compact manifolds, but not in the case of non-compact ones. Even operators on Euclidian spaces are hard to study, for example Laplacian is not Fredholm. Kernels and cokernels of many operators are infinite dimensional Banach spaces, however, they may also at the same time be finitely generated Hilbert modules over some appropriate C^*-algebra. Similarly, orthogonal projections onto kernels and cokernels of many bounded linear operators on Hilbert spaces are not finite rank projections in the classical sense, but they are still finite projections in an appropriate von Neumann algebra. Therefore, many operators that are not semi-Fredholm in the classical sense may become semi-Fredholm in a more general sense if we consider them as operators on Hilbert C^*-modules or as elements of an appropriate von Neumann algebra. Hence, by studying these generalized semi-Fredholm operators, we get a proper extension of the classical semi-Fredholm theory to new classes of operators.

Now, Kečkić and Lazović in [12] established an axiomatic approach to Fredholm theory. They introduced the notion of a finite type element in a unital C^*-algebra which generalizes the notion of the compact operator on the standard Hilbert C^*-module and the notion of a finite operator in a properly infinite von Neumann algebra. They also introduced the notion of a Fredholm type element with respect to the ideal of these finite type elements. This notion is at a same time a generalization of the classical Fredholm operator on a Hilbert space, Fredholm C^*-operator on the standard Hilbert C^*-module defined by Mishchenko and Fomenko and the Fredholm operator on a properly infinite von Neumann algebra defined by Breuer. The index of this Fredholm type element takes values in the K-group. They showed that the set of Fredholm type elements in a unital C^*-algebra is open in the norm topology and they proved a generalization of the Atkinson theorem. Moreover, they proved the multiplicativity of the index in the K-group and that the index is invariant under perturbations of Fredholm type elements by finite type elements.

In this talk we will present the results from [11] regarding semi-Fredholm theory in unital C^*-algebras as a continuation of the approach by Kečkić and Lazović on Fredholm theory in unital C^*-algebras. We will introduce the notion of a semi-Fredholm type element and a semi-Weyl type element with respect to the ideal of finite type elements and obtain a generalization in this setting of several results from the classical semi-Fredholm and semi- Weyl theory of operators on Hilbert spaces. The motivation for this research is not only developing an abstract, axiomatic semi-Fredholm theory in unital C^*-algebras, but also deriving an extension of Breuer‘s Fredholm theory to semi-Fredholm and semi-Weyl theory in properly infinite von Neumann algebras by applying our results to this special case. In the first part of the talk we will present the results in abstract semi-Fredholm theory and semi-Weyl theory in unital C^*-algebras, whereas in the second part of the talk we will focus on the applications of these results to the concrete case of properly infinite von Neumann algebras.

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On some non-associative hyper-algebraic structures

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In this paper, new hyper-algebraic structures called hyperloop, multiloop, polyquasigroup and polyloop, and a special class of polyloop called right Bol polyloop are introduced and studied. It is shown that for any non-commutative (groupoid, quasigroup, loop), commutative and non-commutative (polygroupoid, polyquasigroup, polyloop) can be constructed. It is shown that a right Bol polyloop is characterized by any of seven equivalent identities and has the right alternative properties. Two examples of right Bol loops were constructed with the aid of a ring.

The newly introduced hyper-algebraic structures are:

Definition 1. (Polygroupoid, Polyquasigroup, Polyloop, Multiloop)

Let $\mathcal{M} = (P, \cdot)$ be a polygroupoid. Let $e \in P$ and $\diagup : P \times P \to \mathfrak{P}^*(H)$ and $\diagdown : P \times P \to \mathfrak{P}^*(H)$ such that

(a): (i) $x \in (x \cdot y)\diagup y$ (ii) $x \in (x\diagup y) \cdot y$ (iii) $x \in y\diagdown(y \cdot x)$ (iv) $x \in y \cdot (y\diagdown x)$ for all $x, y \in P$, then $(P, \cdot, \diagdown, \diagup)$ will be called a polyquasigroup.

(b): $x \cdot e = e \cdot x = x$ for all $x \in P$ and $(P, \cdot, \diagdown, \diagup)$ is a polyquasigroup. Then $(P, \cdot, \diagdown, \diagup, e)$ will be called a polyloop.

(c): $x \in x \cdot e = e \cdot x$ for all $x \in P$ and $(P, \cdot, \diagdown, \diagup)$ is a polyquasigroup. Then $(P, \cdot, \diagdown, \diagup, e)$ will be called a multiloop.

(d): $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ for all $x, y, z \in P$ and $(P, \cdot, \diagdown, \diagup)$ is a polyloop. Then $(P, \cdot, \diagdown, \diagup)$ will be called an associative polyloop.

Definition 2. (Right Bol Polyloop)
Let $\mathcal{M} = (P, \cdot, \diagdown, \diagup, e)$ be a polyloop, then $(P, \cdot, \diagdown, \diagup, e)$ will be called a right Bol Polyloop, if it satisfies the identity

$$(xy \cdot z)y = x(yz \cdot y) \ \forall \ x, y, z \in P \tag{1}$$

Result on equivalence between the hyper-algebraic structures in Definition 1 and some existing ones in literature is presented in Theorem 3.

Theorem 3. *Let $(G, \cdot)$ be a polygroupoid.*

(1) *The following are equivalent:*
 (a) *$(G, \cdot)$ is an hyperquasigroup.*
 (b) *$(G, \cdot, \diagdown, \diagup)$ is a polyquasigroup.*
 (c) *$(G, \cdot)$ is an quasigrouphypergroup.*
 (d) *There exist hyperoperations $\diagdown$ and $\diagup$ on G such that $z \in x \cdot y \Leftrightarrow x \in z \diagup y \Leftrightarrow y \in x \diagdown z$ holds for all $x, y, z \in G$.*

(2) *$(G, \cdot, e)$ is a hyperloop if and only if it $(G, \cdot, e)$ is a multiloop.*

(3) *$(G, \cdot)$ is a hypergroup if and only if it is an associative polyquasigroup.*

(4) *$(G, \cdot)$ is an H_v-group if and only if it is a polyquasigroup with WASS.*

(5) *$(G, \cdot)$ is a Marty-Moufang hypergroup (H_m-group) if and only if it is a Moufang polyquasigroup. (Marty-Moufang hypergroup of Bayon and Lygeros [1])*

(6) *$(G, \cdot)$ is a polygroup if and only if it is a associative polyloop.*

Theorem 4 describes a method of construction of commutative and non-commutative polyquasigroups (polyloops) using a non-commutative quasigroup (loop).

Theorem 4. *(Construction of polygroupoid, polyquasigroup and polyloop)*
Given a non-commutative groupoid (quasigroup, loop) $(G, \cdot, \diagdown, \diagup, e)$, define an hyperoperation $\odot : G \times G \to \mathfrak{P}^*(G)$ as $x \odot y = \{xy, yx\}$. Then, there exist left division and right division hyperoperations $\leftthreetimes : G \times G \to \mathfrak{P}^*(G)$ and $\rightthreetimes : G \times G \to \mathfrak{P}^*(G)$ of $\odot$ such that $x \leftthreetimes y = \{x \diagdown y, y \diagup x\}$ and $x \rightthreetimes y = \{x \diagup y, y \diagdown x\}$ respectively and

(1) *$(G, \odot)$ is a commutative polygroupoid.*

(2) *$(G, \odot, \leftthreetimes, \rightthreetimes)$ is a commutative polyquasigroup while $(G, \leftthreetimes, \odot, \leftthreetimes)$ and $(G, \rightthreetimes, \rightthreetimes, \odot)$ are non-commutative polyquasigroups.*

(3) *$(G, \odot, \leftthreetimes, \rightthreetimes, e)$ is a commutative polyloop while $(G, \leftthreetimes, \odot, \leftthreetimes)$ and $(G, \rightthreetimes, \rightthreetimes, \odot)$ are non-commutative polyquasigroups.*

Theorem 5 presents some results on the algebraic properties and characterization of right Bol polyloop as defined by (1) of Definition 2.

Theorem 5. *Let $(P, \cdot, \diagdown, \diagup, e)$ be a polyloop. Then $(P, \cdot, \diagdown, \diagup, e)$ is a right Bol polyloop if and only if either of the following conditions holds:*

(i) $X(yz \cdot y) = (Xy \cdot z)y$

(ii) $x(yZ \cdot y) = (xy \cdot Z)y$
(iii) $x(Yz \cdot Y) = (xY \cdot z)Y$
(iv) $X(yZ \cdot y) = (Xy \cdot Z)y$
(v) $X(Yz \cdot Y) = (XY \cdot z)Y$ *(vi)* $x(YZ \cdot Y) = (xY \cdot Z)Y$
(vi) $X(YZ \cdot Y) = (XY \cdot Z)Y$ *for all* $x, y, z \in P$ *and* $X, Y, Z \subseteq P$.

Example 6. Let $(\mathbb{Z}_2, +, \cdot)$ be the ring of integer modulo 2 and let $G = \mathbb{Z}_2^3$. For (i, j, k) and (p, q, r) in G, define

$$(i, j, k) * (p, q, r) = (i + p, j + q, k + r + jpq).$$

Consider $\mathbb{Z}_2^3//N \subseteq P(\mathbb{Z}_2^3)$ where $N = N(\mathbb{Z}_2^3, *) = \{(0,0,0), (0,1,0), (1,0,0), (0,1,1)\}$ is the nucleus of $(\mathbb{Z}_2^3, *)$ so that

$$\mathbb{Z}_2^3//N = \Big\{ \big\{ (i,j,k), (i,j+1,k), (i,j,k+1), (i+1,j,k), (i,j+1,k+1) \big\} \;\big|\; i,j,k \in \mathbb{Z}_2 \Big\}.$$

Define an hyperoperation $'\circ'$ on $\mathbb{Z}_2^3//N$ as follows

$$(i,j,k)N \circ (p,q,r)N = \Big\{ \Big\{ \big(i+a+p, j+b+q, k+c+jab+r+(j+b)pq\big),$$
$$\big(i+a+p, j+b+q+1, k+c+jab+r+(j+b)pq\big), \big(i+a+p, j+b+q, k+c+jab+r+$$
$$(j+b)pq+1\big), \big(i+a+p+1, j+b+q, k+c+jab+k+(j+b)pq\big),$$
$$\big(i+a+p, j+b+q+1, k+c+jab+r+(j+b)pq+1\big) \Big\} \;\Big|\; i,j,k,p,q,r \in \mathbb{Z}_2, a,b,c \in N \Big\}.$$

Then, $\Big(\mathbb{Z}_2^3//N, \circ\Big)$ is a right Bol polyloop.

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The rank of Mordell-Weil groups of surfaces

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Let $S \longrightarrow C$ be a fibraton of surface, and we can define Mordell-Weil groups. In fact, they are Abelian groups. In 1989, Prof. Mok raised the following question in [1]:

Problem 1. How to determine the rank of Mordell-Weil group >0?

In [2] and [4], the authors discuss the above problem. In this talk, we try to give some new views in this problem. Especially, we use the number of singular fibers to determine whether the rank is zero or not.

Theorem 2. *Let* $S \to \mathbb{P}^1$ *be a fibraton of surface. If* $s_1 > 4g$*, then the rank of Mordell-Weil group* > 0*, where* s_1 *is the number of fiber whose Jacobian is singular.*

We will also discuss the following similar problem in this talk.

Problem 3. How to determine the Mordell-Weil group is trivial or not?

Prof. Kitagawa and Prof. Konno used the pencils of surfaces to consider this problem in [3]. Here, we give the following theorem for elliptic fibrations in another way.

Theorem 4. *Let $S \longrightarrow \mathbb{P}^1$ be an elliptic fibration of surface with s singular fibers. If $s > 3$, then Mordell-weil group is not trivial.*

For the above two problems, our results are the best. Because we have the following example:

Example 5. The Weiestrass equation $y^2 = x^3 - t^4x + t^5$ corresponds to an ellptic fibration over $\mathbb{P}^1$ with II^*, I_1 and I_1 at $t = 0,\ t = \pm\frac{3\sqrt{3}}{2}$. It is easy to see that Trivial lattice is E_8, and Mordell-Weil group is trivial.

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On Asplund spaces $C_k(X)$ with the compact-open topology

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Recall that a Banach space E is called an Asplund space if every separable Banach subspace of E has separable dual. A celebrated theorem of Namioka and Phelps says that for a compact space X, the Banach space $C(X)$ of continuous real-valued functions on X is Asplund if and only if X is scattered. We extend this result to the class of spaces $C_k(X)$ of continuous real-valued functions endowed with the compact-open topology for several natural classes of non-compact Tychonoff spaces X. The concept of Δ_1-spaces recently introduced and studied has been shown to be applicable for this research.

Explicit construction of explicit real algebraic functions and real algebraic manifolds via Reeb graphs

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In this talk, we present explicit real algebraic functions on explicit real manifolds via *Reeb graphs*. Our study is mainly motivated by real algebraic geometry, pioneered by Nash and Tognoli for example. A smooth closed manifold can be regarded as a non-singular real algebraic manifold. Existence theory of real algebraic manifolds and real algebraic maps has been also well-known. We can easily attain such objects and morphisms whereas it is very difficult to know explicitly. For more precise exposition, see [4] for example.

Of course some specific exmaples of real algebraic maps are well-known. Canonical embeddings and projections of unit spheres are simplest examples. As functions which are in considerable cases regarded as generalized ones, Lie groups and so-called *symmetric spaces* have nice functions represented as real polynomial functions. See [7] for example. However, it is difficult to know their global structures and properties explicitly in general.

Problem 1. Can we know global structures and properties of the functions and maps. For example, can we know information on preimages?

For this, we consider the following problem, established in [10]. This comes from singularity theory of smooth maps and applications to differential topology of manifolds. The *Reeb graph* of a smooth function is the graph whose underlying space is the natural quotient space of the manifold and consists of all connected components of preimages. Its vertex set is the set of all connected components containing some singular points of the function. As [9] shows, for smooth functions with finitely many singular values, we can have such graphs. [8] is a pioneering paper on this notion. Reeb graphs have some information of the manifolds nicely and fundamental and strong tools in geometry of manifolds.

Problem 2. Can we reconstruct a nice smooth function on some manifold whose Reeb graph is the given graph? We do not fix the manifold beforehand.

[10] constructs desired functions on closed surfaces for some nice graphs. [5] extends this to arbitrary finite graphs. [6] considers such a problem for a certain class of finite graphs and Morse functions such that connected components of preimages having no singular points are spheres. Our study [1] considers the following problem first. It is for functions on 3-dimensional closed manifolds. [9] presents a related general result through our informal discussions on [1].

Problem 3. Can we construct the function in Problem 2 with prescribed preimages?

This talk is on answers to the following problem, pioneered by the speaker first in [2].

Problem 4. Can we construct these functions and the manifolds in finer categories such as the real analytic category and the real algebraic category, for example?

We present our main results with several notions we need. An *algebraic domain* D is a bounded open set in the real affine space $\mathbb{R}^k$ surrounded by finitely many mutually disjoint non-singular connected real algebraic hypersurfaces each S_j of which is the zero set of some real polynomial f_j. The *Poincaré-Reeb graph* of it is a canonically obtained graph whose underlying space is the natural quotient space of the closure $\overline{D}$ of the domain and consists of all connected components of preimages for the restriction of the projection $\pi(x_1, \cdots, x_k) := x_1$ to $\overline{D}$. Its vertex set is defined as the set of all connected components containing some singular points of the function $\pi|_{\overline{D}-D}$.

Theorem 5 ([2])**.** *Let G be a Poincaré-Reeb graph of $D \subset \mathbb{R}^k$. Let D be an algebraic domain represented as the intersection $\bigcap\{x \in \mathbb{R}^k \mid f_j(x) > 0\}$. Then we can construct a smooth real algebraic function whose Reeb graph is isomorphic to G on some non-singular real algebraic closed manifold.*

Example 6. Any Poincaré-Reeb graph G of any bounded connected open set $D \subset \mathbb{R}^k$ surrounded by finitely many mutually disjoint spheres of fixed radii satisfies the assumption of Theorem 5. See also FIGURE 1 of [2].

In the proof, first we construct a nice smooth real algebraic map into $\mathbb{R}^k$ whose image is $\overline{D}$. More precisely, we construct one such that the preimage of a point in the boundary is a one-point set and that the preimage of a point in the interior is a sphere. Last we compose the projection.

Theorem 7 ([3])**.** *Let $l > 3$ and $m > 2$ be integers. Let $\{t_j\}_{j=1}^{l}$ be an increasing sequence of real numbers. Let $\{F_j\}_{j=1}^{l-1}$ be a family of smooth manifolds satisfying the following conditions.*

- *F_1 and F_{l-1} are diffeomorphic to the $(m-1)$-dimensional unit spheres S^{m-1}.*
- *The others are diffeomorphic to S^{m-1} or represented as connected sums of finitely many manifolds diffeomorphic to the products $S^j \times S^{m-j-1}$ for some integers $1 \leq j \leq m-2$: the connected sum is taken in the smooth category. For adjacent integers $1 \leq j \leq l-2$ and $j+1$, either F_j or F_{j+1} is not diffeomorphic to the unit sphere.*

Then we have an m-dimensional non-singular real algebraic closed and connected manifold M and a smooth real algebraic function $f : M \to \mathbb{R}$ such that the number of singular points is finite, that $\{t_j\}_{j=1}^{l}$ is the set of all singular values and that the preimage $f^{-1}(p_j)$ is diffeomorphic to F_j for $p_j \in (t_j, t_{j+1})$.

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Conformal equivalence of 3-webs

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Let 3-web $W_3\left<\omega_1, w_2, \omega_3\right>$ defined in a domain D on the conformal plane $(\mathbb{R}^2, g)$. We say that this 3-web is *regular* in D if in this domain:

(1) The discriminant
$$\widetilde{\Delta} = -4I_2 + I_1^2 + 18I_1I_2 - 4I_1^3 - 27I_2^3$$
differs from zero.

(2) Invariants
$$I_1 = \frac{J_2}{J_1^2} \quad and \quad I_2 = \frac{J_3}{J_1^3}$$
are functionally independent in the domain, that is, the differential 2-form $\Omega = dI_1 \wedge dI_2 \neq 0$.
Moreover, invariants I_1, I_2 are coordinates in the domain.

We remark that the elementary symmetric functions
$$\begin{aligned} J_1 &= \lambda_1 + \lambda_2 + \lambda_3, \\ J_2 &= \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3, \\ J_3 &= \lambda_1\lambda_2\lambda_3 \end{aligned}$$
are S_3 - invariants and $\lambda_1, \lambda_2, \lambda_3$ are positive smooth functions.

Let's number now forms ω_1, w_2, ω_3 in the domain and say that the 3-web is *oriented* in the domain if in this numbering $\omega_1 \wedge \omega_2 = r_{12}\Omega$, where $r_{12} > 0$. In this case we'll scale forms ω_i in such a way, that
$$\omega_1 \wedge \omega_2 = \Omega.$$
In opposite case, we call the 3-web *non-oriented* and scale the 1-forms ω_i in such a way, that
$$\omega_1 \wedge \omega_2 = -\Omega.$$

In these both cases we decompose 1-forms ω_i in the invariant coordinates I_1, I_2

$$\omega_i = \sum_{j=1}^{2} w_{ij} dI_j,$$

Then, all functions $w_{ij}, i = 1,2,3; j = 1,2$ are conformal invariants, satisfying the following additional relations

$$\sum_{i=1}^{3} w_{ij} = 0, \ \ j = 1,2.$$

Now, let's write down the standard metric tensor g in invariant coordinates as follows

$$g = \sum_{i,j=1}^{2} g_{ij} dI_i \otimes dI_j.$$

Remark, that the volume 2-form Ω_g, associated with metric g, is the following

$$\Omega_g = \sqrt{\det \|g_{ij}\|} dI_1 \wedge dI_2.$$

Therefore, the metric tensor

$$\widetilde{g} = \frac{g}{\sqrt{\det \|g_{ij}\|}}$$

has the associated volume form $\Omega_{\widetilde{g}} = \Omega$.

Finally, we get the following result.

Theorem 1. *Let 3-web $W_3 \langle \omega_1, w_2, \omega_3 \rangle$ be regular in a domain D in the conformal plane $(\mathbb{R}^2, g)$. Then the above functions*

$$w_{ij}, \qquad \widetilde{g}_{ij} = \frac{g_{ij}}{\sqrt{\det \|g_{ij}\|}}$$

that are components of 1-forms ω_1, ω_2 and the metric tensor $\widetilde{g}$ in the invariant coordinates I_1, I_2, are conformal invariants of plane 3-webs.

Moreover, any two regular 3-webs are conformly equivalent if and only if the corresponding functions w_{ij} and $\widetilde{g}_{ij}$ coinside.

The fundamental group of Riemann surface via Riemann's existence theorem

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One of the classical things we learn in any complex analysis course is the structure of the fundamental group of Riemann surfaces that it is given by the following theorem:

Theorem 1. *The fundamental group of Riemann Surfaces of genus g is given by $2g$ generators with one relation :*

$$\prod_{i=1}^{g} [a_i, b_i] = 1 \tag{1}$$

$[a_i, b_i]$ *is the commutator of* 2 *group elements given by:* $[x, y] = xy(yx)^{-1}$

However when you first encounter Algebraic curves (Riemann Surfaces) they are presented through cuts and analytic continuation in a pictersque way. I have never seen a proof in the literature that the fundamental group of the surface given pictorially by cuts has a representation given by the theorem. Indeed the starting point of surface groups is the commutation relation. In this talk I will try to fill this gap. While I don't have a formal proof yet I will present some results that to me seems somewhat surprising. The talk is elementary in nature and no knowledge of heavy topology is required.

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Problem with integral conditions for evolution equations in Banach space

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Let A be a given linear operator acting in the Banach space B, and for this operator, arbitrary powers $A^n : B \to B$, $n \in \mathbb{N}$. Denote be $x(\lambda)$ the eigenvector of the operator A which corresponds to its eigenvalue $\lambda \in \Lambda$, i.s. nonzero solution in B of the equation $Ax(\lambda) = \lambda x(\lambda), \lambda \in \Lambda$, where $\lambda \subset \mathbb{C}$. If Λ is not an eigenvalue of the operator A then $x(\lambda) = 0$.

We consider next problem with integrals condition

$$\frac{d^2U}{dt^2} + a(A)\frac{dU}{dt} + b(A)U = 0, \quad t \in [0, T], \tag{1}$$

$$\int_0^T U(t)dt = \varphi_1, \quad \int_0^T tU(t)dt = \varphi_2, \tag{2}$$

where $\varphi_1, \varphi_2 \in B$, $T > 0$, $u : (0; \alpha) \cup (\beta; h) \to B$ - is an unknown function, $a(A) : B \to B$, $b(A) : B \to B$ - is abstract operators with entire symbols $a(\lambda) \neq const$, $b(\lambda) \neq const$.

Let for $m = \{0, 1\}$ function $M_m(t, \lambda)$ be a solution of the problem

$$\frac{d^2M_m(t,\lambda)}{dt^2} + a(\lambda)\frac{dM_m(t,\lambda)}{dt} + b(\lambda)M_m(t,\lambda) = 0, \quad t \in [0, T], \tag{3}$$

$$\int_0^T t^k M_m(t,\lambda)dt = \delta_{km}, \quad k = \{0, 1\}, \tag{4}$$

where δ_{km} is the Kronecker symbol.

Definition. We shall say that vectors $\varphi_1, \varphi_2 \in B$, from B belong $L \subset B$. If dependent exists on linear operators $R_{\varphi_k}(\lambda) : B \to B$, $\lambda \in \Lambda$ and measures μ_{φ_k} such that

$$\varphi_k = \int_\Lambda R_{\varphi_k}(\lambda)x(\lambda)d\mu_{\varphi_k(\lambda)}. \tag{5}$$

Theorem. Let in the problem (1), (2), the vectors φ_k belongs L. There $\varphi_k, k = \{1, 2\}$ can be represented in the form (5). Then the formula

$$U(t) = \int_{\Lambda} R_{\varphi_1}(\lambda)\{M_0(t,\lambda)x(\lambda)d\mu_{\varphi_1}(\lambda) + \int_{\Lambda} R_{\varphi_2}(\lambda)\{M_1(t,\lambda)x(\lambda)d\mu_{\varphi_2}(\lambda),$$

defines solution of the problem (1), (2), $M_m(t, \lambda)$ is a solution of the problem (3), (4).

Be means of the differential-symbol method [5] we construct of the problem (1), (2).

Solution of the problem (3), (4) according to the differential-symbol [1, 2] method exists and uniquess in the class of quasi-polynomials.

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Deformational symmetries of functions with isolated singularities on the Mobius band

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Let M be a smooth compact 2-dimensional manifold which have a non-empty boundary, and P be either a real line or a circle. Denote by $D(M, Y)$ the group of diffeomorphisms of M fixed on a closed subset $Y \subset M$. There is a natural right action of the group $D(M, Y)$ on the space of smooth functions $C^\infty(M, \mathbb{R})$ defined by the following rule: $(h, f) \longmapsto f \circ h$, where $h \in D(M, Y)$, $f \in C^\infty(M, \mathbb{R})$.

Let

$$\mathcal{O}(f, Y) = \{f \circ h \mid h \in \mathcal{D}(M, Y)\}$$

be the *orbit* of f under this action. Endow $C^\infty(M, \mathbb{R})$ with Whitney C^∞-topology and $\mathcal{O}(f, Y)$ with induced one.

Definition 1. Denote by $\mathcal{F}(M, P)$ the space of smooth maps $f \in C^\infty(M, P)$ having the following properties:

(1) the map f takes constant values at each connected component of ∂M and has no critical points on it;
(2) for every critical point z of f there is a local presentation $f_z : \mathbb{R}^2 \to \mathbb{R}$ of f near z such that f_z is a homogeneous polynomial $\mathbb{R}^2 \to \mathbb{R}$ without multiple factors.

Definition 2. Let G, H be groups, $m \in \mathbb{Z}$ and $\gamma\colon H \to H$ be automorphism of order 2. Define the automorphism $\phi\colon G^{2m} \times H^m \to G^{2m} \times H^m$ by the formula

$$\phi(g_0, \ldots, g_{2m-1}, h_0, \ldots, h_{m-1}) = (g_{2m-1}, g_0, \ldots, g_{2m-2}, h_1, h_2, \ldots, h_{m-1}, \gamma(h_0)).$$

This automorphism ϕ generates homomorphism $\phi'\colon \mathbb{Z} \to G^{2m} \times H^m$. The corresponding semidirect product $G^{2m} \times H^m \rtimes_{\phi'} \mathbb{Z}$ will be denoted $(G, H) \wr_{\gamma,m} \mathbb{Z}$.

Definition 3. Let $\mathcal{P}$ be a minimal class of groups satisfying the following conditions:

1) $1 \in \mathcal{P}$;
2) if $A, B \in \mathcal{P}$, then $A \times B \in \mathcal{P}$;
3) if $A \in \mathcal{P}$ and $n \geq 1$, then $A \wr_n \mathbb{Z} \in \mathcal{P}$.

It was shown in [2] that if M has negative Euler characterictic, then fundamental groups of orbits of functions in $\mathcal{F}(M, P)$ are direct products of such groups for functions only on cylinders, disks and Möbius bands. Moreover, if M is either a 2-disk or a cylinder, then $\pi_1\mathcal{O}(f, \partial M) \in \mathcal{P}$.

Theorem 4. *Let M be a Möbius band and let $f \in \mathcal{F}(M, P)$. Then*

$$\pi_1\mathcal{O}(f, \partial M) \cong A \times (G, H) \wr_{\gamma,m} \mathbb{Z}, \text{ where } A, G, H \in \mathcal{P}.$$

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Codes from zero-divisor super-λ graph

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In coding theory, super-λ graphs were used to build linear codes. Thus, in order to see whether the zero-divisor graphs might be useful into this context, it is natural to study when zero-divisor graphs of some non elementary ring constructions are super-λ graphs. In this presentation, we show that there are various classes of rings whose zero-divisor graphs are super-λ. We apply these results to determine parameters of some linear codes associated to zero-divisor graphs.

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Twisted Sasaki metric on the unit tangent bundle and harmonicity

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Let (M^n, g) be an n-dimensional Riemannian manifold, TM^n be its tangent bundle, $\mathfrak{X}(M^n)$ be the Lie algebra of smooth vector fields of a Riemannian manifold (M^n, g), ∇ be the Levi-Civita connection on M^n. The standard metric on the tangent bundle of Riemannian manifold (M^n, g) is the Sasaki metric [7]. It can be completely defined by scalar products of various combinations of vertical and horizontal lifts of vector fields. The Sasaki metric weakly inherits the base manifold properties. That is why the rigidity of Sasaki metric motivates many authors consider various deformations of Sasaki metric (see [1], [3] and others).

Belarbi L. and El Hendi H. introduce in [2] the twisted Sasaki metric on the tangent bundle TM as a new natural metric non-rigid on TM. The twisted Sasaki metric is defined as follows.

Definition 1. [2] Let (M^n, g) be a Riemannian manifold and $\delta, \varepsilon : M^n \longrightarrow \mathbb{R}$ be strictly positive smooth functions. On the tangent bundle TM^n, we define a twisted Sasaki metric noted $G^{\delta,\varepsilon}$ by

$$\begin{aligned} G^{\delta,\varepsilon}_{(x,\xi)}(X^h, Y^h) &= \delta(x) g_x(X, Y), \\ G^{\delta,\varepsilon}_{(x,\xi)}(X^h, Y^v) &= 0, \\ G^{\delta,\varepsilon}_{(x,\xi)}(X^v, Y^v) &= \varepsilon(x) g_x(X, Y). \end{aligned}$$

for all vector fields $X, Y \in \mathfrak{X}(M^n)$ and $(x, \xi) \in TM^n$.

Note that, if $\delta = \varepsilon = 1$, then $G^{\delta,\varepsilon}$ is the Sasaki metric [7]. If $\delta = 1$, then $G^{\delta,\varepsilon}$ is the vertical rescaled metric (see [3], [4]).

For a unit vector field ξ on a compact Riemannian manifold (M, g), Gerrit Weigmink [8] considered a very natural geometric functional

$$\int_M ||A_\xi||^2 dVol(M),$$

where $||A_\xi||$ is a norm of the Nomizu operator $A_\xi X = -\nabla_X \xi$, i.e. $||A_\xi||^2 = \sum_{i=1}^n g(A_\xi e_i, A_\xi e_i)$ relative to some orthonormal frame $(e_1, \ldots, e_n)$. It was proved, that this functional is unbounded from above. The critical points of this functional was called *harmonic unit vector fields*. G. Wigmink proved, that a unit vector field ξ on compact Riemannian manifold is harmonic if and only if

$$\bar{\Delta}\xi = ||A_\xi||^2 \xi,$$

where $\bar{\Delta}\xi$ is rough Laplacian (or Bochner Laplacian) of the field ξ defined as $\bar{\Delta}\xi = -trace \nabla^2 \xi$, where $\nabla^2_{X,Y} = \nabla_X \nabla_Y - \nabla_{\nabla_X Y}$.

On the other hand (see [5]), the energy of a *mapping* $\phi : (M^n, g) \to (N^k, h)$ between Riemannian manifolds is defined as

$$E(\phi) := \frac{1}{2} \int_M |d\phi|^2 dVol_M.$$

The mapping ϕ *is called harmonic* if it the critical point of the energy functional. It was proved that the mapping ϕ is harmonic if and only if the divergence of its differential vanishes, or equivalently its tension field $\tau(\phi) = div(d\phi)$ vanishes identically, where $|d\phi|$ is a norm of 1-form $d\phi$ in the cotangent bundle T^*M^n. Supposing on T_1M^n the Sasaki metric g_S, a unit vector field ξ as a mapping $\xi : (M^n, g) \to (T_1M^n, g_S)$ *defines a harmonic map* if and only if it is *harmonic* and, in addition, $\sum_{i=1}^n R(\xi, A_\xi e_i)e_i = 0$ relative to some othonormal frame $\{e_i\}_{i=1}^n$.

In the present research we define the twisted Sasaki metric [2] on the unit tangent bundle T_1M^n of n-dimensional Riemannian manifold (M^n, g), obtain Kowalski-type formulas, calculate the tension field of the mapping ξ: $M^n \to (T_1M^n, G^{\delta,\varepsilon})$. As a main result, for twisted Sasaki metric $G^{\delta,\varepsilon}$ on the unit tangent bundle T_1M^n of n-dimensional Riemannian manifold (M^n, g) we obtain the necessary and sufficient conditions for harmonicity of left-invariant unit vector field ξ and mapping ξ: $M^n \to (T_1M^n, G^{\delta,\varepsilon})$.

Theorem 2. *Unit vector field ξ is harmonic on n-dimensional Riemannian manifold (M^n, g) equipped with twisted Sasaki metric $G^{\delta,\varepsilon}$ on the unit tangent bundle T_1M^n if and only if*

$$\bar{\Delta}\xi + \frac{1}{\varepsilon}A_\xi(\nabla\varepsilon) = ||A_\xi||^2\xi. \tag{1}$$

Harmonic unit vector field ξ defines a harmonic mapping ξ: $M^n \to (T_1M^n, G^{\delta,\varepsilon})$ on n-dimensional Riemannian manifold (M^n, g) equipped with twisted Sasaki metric $G^{\delta,\varepsilon}$ on the unit tangent bundle T_1M^n if and only if

$$2\varepsilon \cdot trace(Hm_\xi) + (n-2)\nabla\delta + ||A_\xi||^2\nabla\varepsilon = 0. \tag{2}$$

As an examples, we consider the necessary and sufficient conditions for harmonicity of left-invariant unit vector field ξ and harmonic mapping ξ: $M^3 \to (T_1M^3, G^{\delta,\varepsilon})$ on 3-dimensional unimodular Lie group equipped with twisted Sasaki metric on the unit tangent bundle T_1M^3, using orthonormal frame of Milnor J. [6]. In addition, we consider some examples of deformations that preserves existence harmonic left-invariant unit vector fields ξ and harmonic mapping ξ: $M^3 \to (T_1M^3, G^{\delta,\varepsilon})$ on 3-dimensional unimodular Lie groups with the left invariant metric.

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Lie structures of the Sheffer group over a Hilbert space

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Umbral calculus (also called calculus of finite differences) is essentially the theory of Sheffer polynomial sequences, which are characterised by the exponential form of their generating function. The class of Sheffer sequences includes the binomial sequences and Appell sequences. After a long period when one-dimensional umbral calculus was used for purely formal calculations, the theory became rigorous in the 1970s due to the seminal works of G.-C. Rota, S. Roman and their co-authors. Their theory is nowadays called the modern umbral calculus, see e.g. the monographs [4, 8]. Umbral calculus found applications in combinatorics, theory of special functions, approximation theory, probability and statistics, topology and physics, see e.g. the survey paper [2] for a long list of references. A central object of studies of umbral calculus is the umbral composition, which equips the set of all Sheffer sequences with a group structure. This group is isomorphic to the Riordan group of infinite lower triangular matrices [6, 10]. Recently, Cheon et al. [3] (see also Bacher [1]) introduced a Lie group structure on the Riordan group and found the corresponding Lie algebra.

A lot of research has been done to extend the classical umbral calculus to the multivariate case, see Section 4 in [2] for a list of references. However, this research had a significant drawback of being basis-dependent. The paper [5] developed foundations of infinite-dimensional, basis-independent umbral calculus.

In this talk, we will discuss Lie structures of the group of Sheffer polynomials over a Hilbert space. Let

$$\mathcal{H}_+ \subset \mathcal{H}_0 \subset \mathcal{H}_0$$

be standard triple of real separable Hilbert spaces, i.e., the Hilbert space $\mathcal{H}_+$ is densely and continuously embedded into $\mathcal{H}_0$ and $\mathcal{H}_-$ is the dual of $\mathcal{H}_+$, while the dual paring between elements of $\mathcal{H}_-$ and $\mathcal{H}_+$ is determined by the inner product in $\mathcal{H}_0$. Then, for each n, we also get a standard triple

$$\mathcal{H}_+^{\odot n} \subset \mathcal{H}_0^{\odot n} \subset \mathcal{H}_0^{\odot n}.$$

Here $\odot$ denotes the symmetric tensor product. For $F^{(n)} \in \mathcal{H}_-^{\odot n}$ and $f^{(n)} \in \mathcal{H}_+^{\odot n}$, we denote by $\langle F^{(n)}, f^{(n)}\rangle$ the dual pairing between $F^{(n)}$ and $f^{(n)}$. (For a real Hilbert space $\mathcal{H}$, we define $\mathcal{H}^{\odot 0} := \mathbb{R}$.)

A (continuous) polynomial on $\mathcal{H}_-$ is a function $p : \mathcal{H}_- \to \mathbb{R}$ of the form

$$p(\omega) = \sum_{i=0}^{n} \langle \omega^{\odot i}, f^{(i)}\rangle, \quad \omega \in \mathcal{H}_-,\ f^{(i)} \in \mathcal{H}_+^{\odot i}, \quad i = 0, 1, \ldots, n,\ n \in \mathbb{N}_0. \tag{1}$$

We denote by $\mathcal{P}(\mathcal{H}_-)$ the vector space of all polynomials on $\mathcal{H}_-$. By identifying the polynomial $p(\omega)$ in (1) with the sequence $(f^{(i)})$, we endow $\mathcal{P}(\mathcal{H}_-)$ with the topology of the topological direct sum of the Hilbert spaces $\mathcal{H}_+^{\odot i}$, $i \in \mathbb{N}_0$.

A monic polynomial sequence on $\mathcal{H}_-$ is a continuous linear map $P \in \mathcal{L}(\mathcal{P}(\mathcal{H}_-))$ that satisfies

$$(P\langle \cdot^{\odot n}, f^{(n)}\rangle)(\omega) = \sum_{i=0}^{n} \langle \omega^{\odot i}, p_{in} f^{(n)}\rangle, \tag{2}$$

where $p_{in} \in \mathcal{L}(\mathcal{H}_+^{\odot n}, \mathcal{H}_+^{\odot i})$ and $p_{nn} = \mathbf{1}$. Denote by $p_{in}^* \in \mathcal{L}(\mathcal{H}_+^{\odot i}, \mathcal{H}_+^{\odot n})$ the adjoint (dual) operator of p_{in}. Then

$$(P\langle \cdot^{\odot n}, f^{(n)}\rangle)(\omega) = \langle p^{(n)}(\omega), f^{(n)}\rangle,$$

where $p^{(n)}(\omega) \in \mathcal{H}_-^{\odot n}$ is given by $p^{(n)}(\omega) := \sum_{i=0}^n p_{in}^* \omega^{\odot i}$. Thus, $p^{(n)} : \mathcal{H}_- \to \mathcal{H}_-^{\odot n}$, and we can identify the linear operator P from (2) with the sequence $(p^{(n)})_{n=0}^\infty$.

A monic polynomial sequence $(p^{(n)})_{n=0}^\infty$ is called a Sheffer sequence (on $\mathcal{H}_-$) if it has the exponential generating function of the form

$$\sum_{n=0}^\infty \frac{1}{n!}\langle p^{(n)}(\omega), \xi^{\odot n}\rangle = \exp\left[\langle \omega, B(\xi)\rangle\right] A(\xi), \quad \omega \in \mathcal{H}_-,\ \xi \in \mathcal{H}_+, \tag{3}$$

where $B(\xi) = \xi + \sum_{k=2}^\infty b_k \xi^{\odot k}$, $b_k \in \mathcal{L}(\mathcal{H}_+^{\odot k}, \mathcal{H}_+)$, $A(\xi) = 1 + \sum_{k=1}^\infty a_k \xi^{\odot k}$, $a_k \in \mathcal{L}(\mathcal{H}_+^{\odot k}, \mathbb{R})$, and the equality (3) is understood as the equality of formal tensor power series in ξ, see [5]. We denote by $\mathbb{S}$ the set of all Sheffer sequences on $\mathcal{H}_-$. We also denote by $\mathbb{A}$ the set of all Appell sequences, i.e., the Sheffer sequences for which $B(\xi) = \xi$ in (3), and we denote by $\mathbb{B}$ the set of all the binomial sequences, i.e., the Sheffer sequences for which $A(\xi) = 1$ in (3).

Since elements of $\mathbb{S}$ were defined through continuous linear operators in $\mathcal{P}(\mathcal{H}_-)$, one can ask a natural question whether a product of two such operators yields a Sheffer sequence. The answer to this question is positive, and furthermore the set $\mathbb{S}$, equipped with this product, becomes a group. Note that the neutral element in this group is the identity operator, equivalently the monomial sequence $p^{(n)}(\omega) = \omega^{\odot n}$. Furthermore, both $\mathbb{A}$ and $\mathbb{B}$ are subgroups $\mathbb{S}$, $\mathbb{A}$ is a normal subgroup of $\mathbb{S}$, and the Sheffer group $\mathbb{S}$ is a semidirect product of the Appell group $\mathbb{A}$ and the binomial group $\mathbb{B}$.

In the talk, we will discuss the following results:

- We will show that $\mathbb{S}$, $\mathbb{A}$ and $\mathbb{B}$ can be described as infinite-dimensional Lie groups, in the sense of Milnor [7], see also [9, Chapter 3].
- We will find the explicit form of the Lie algebra of each of these Lie groups, and we will find a Lie bracket on them.
- We will conclude that the Sheffer group is constructed from two basic operations: gradient of polynomials on $\mathcal{H}_-$ and multiplication by ω.

This is joint result with Dmitri Finkelshtein (Swansea University) and Maria João Oliveira (Universidade Aberta, Lisbon).

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The Gorenstein flat model structure relative to a semidualizing module

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Abstract.
A model structure on a category is a formal way of introducing a homotopy theory on that category, and if the model structure is abelian and hereditary, its homotopy category is known to be triangulated. So a good way to both build and model a triangulated category is to build a hereditary abelian model structure.

Let R be a ring and C be a left R-module. In this talk, we construct a unique hereditary abelian model structure on the category of left R-modules, in which the cofibrations are the monomorphisms with $\mathrm{G_C}$-flat cokernel and the fibrations are the epimorphisms with $\mathcal{C}_C$-cotorsion kernel belonging to the Bass class $\mathcal{B}_C(R)$.

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On the structure of the distribution of one random series.

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Let $s \in N, s > 1, \sum_{n=1}^{+\infty} a_n$— convergent series, ξ_n — a sequence of independent random variables that acquire the values $0 < a_{0n} < a_{1n} < ... < a_{(s-1)n} < 1$ with probabilities $p_{0n}, p_{1n}, ..., p_{(s-1)n}$ respectively. Consider a random variable

$$\xi = \sum_{n=1}^{+\infty} a_n \xi_n.$$

According to the Jessen-Wintner theorem [1], the distribution ξ is pure. Partial cases for the ξ distribution were considered in the works of [2], [3], [4].

Let

$$M = \{\sum_{n=1}^{+\infty} b_n a_n | b_n \in \{a_{0n}; a_{1n}; ...; a_{(s-1)n}\} \forall n \in N\}.$$

Theorem 1. *Let the sequence $(s^n|a_n|)$ be bounded.*

If $\lambda(M) = 0$, then the distribution ξ is discrete if and only if

$$\prod_{n=1}^{+\infty} \max\{p_{0n}; p_{1n}; ...; p_{(s-1)n}\} = 0,$$

singular if and only if

$$\prod_{n=1}^{+\infty} \max\{p_{0n}; p_{1n}; ...; p_{(s-1)n}\} > 0.$$

If $\lambda(M) > 0$, then the distribution ξ is discrete if and only if

$$\prod_{n=1}^{+\infty} \max\{p_{0n}; p_{1n}; ...; p_{(s-1)n}\} = 0,$$

absolutely continuous if and only if

$$\sum_{n=1}^{+\infty} \sum_{j=0}^{s-1} (p_{jn} - \frac{1}{s})^2 < +\infty,$$

singular if and only if

$$\sum_{n=1}^{+\infty} \sum_{j=0}^{s-1} (p_{jn} - \frac{1}{s})^2 = +\infty.$$

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Homotopy types of diffeomorphisms groups of simplest Morse-Bott foliations on lens spaces

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Let F be the Morse-Bott foliation on the solid torus $T = S^1 \times D^2$ into 2-tori parallel to the boundary and one singular circle $S^1 \times 0$. A diffeomorphism $h : T \to T$ is called *foliated* (resp. *leaf preserving*) if for each leaf $\omega \in F$ its image $h(\omega)$ is also leaf of F (resp. $h(\omega) = \omega$). Gluing two copies of T by some diffeomorphism between their boundaries, one gets a lens space $L_{p,q}$ with a Morse-Bott foliation $F_{p,q}$ obtained from F on each copy of T. Denote by $\mathcal{D}^{fol}(T, \partial T)$ and $\mathcal{D}^{lp}(T, \partial T)$ respectively the groups of foliated and leaf preserving diffeomorphisms of T fixed on the boundary ∂T. Similarly, let $\mathcal{D}^{fol}(L_{p,q})$ and $\mathcal{D}^{lp}(L_{p,q})$ be respectively the groups of foliated and leaf preserving diffeomorphisms of $F_{p,q}$. Endow all those groups with the corresponding C^∞ Whitney topologies. The aim of the talk is give a complete description the homotopy types of the above groups $\mathcal{D}^{fol}(T, \partial T)$, $\mathcal{D}^{lp}(T, \partial T)$, $\mathcal{D}^{fol}(L_{p,q})$, $\mathcal{D}^{lp}(L_{p,q})$ for all p, q.

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Spaces of idempotent measures with countable support

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Methods of infinite-dimensional topology can be applied to the problem of description of topology of various objects in particular, hyperspaces and spaces of probability measures (see [1]-[3]). It is our aim to consider the topology of spaces of idempotent measures, which are counterparts of probability measures in the idempotent mathematics (see, e.g, [5]).

Having in mind the identification of every idempotent measure with its density function, we consider, for every metric space X, the set $\bar{I}(X)$ of the closed subsets A of $X \times [0, 1]$ satisfying the following properties:

Definition 1. (1) A is saturated, i.e. $\forall (x, t) \in A\ \forall t',\ 0 \le t' \le t \Rightarrow (x, t') \in A$;
(2) $X \times \{0\} \subset A$;
(3) $A \cap (X \times \{1\}) \neq$.

The support of any $A \in \bar{I}(X)$ is the set

$$\text{supp}(A) = \text{Cl}\ (\{\ x \in X \mid \exists t > 0, (x,t) \in A\}).$$

The set $\in \bar{I}(X)$ is endowed with the Hausdorff metric induced by the max-metric on $X \times [0,1]$. Some hyperspaces of countable closed sets in metric spaces are considered in [3]. Denote by A' is the derived set (the set of all accumulation points) of A.

We introduce the following spaces of idempotent measures:

$$\bar{A}_n(X) = \{A \in \bar{I}(X) \mid 1 \leq |(\text{supp}(A))'| \leq n\} \quad (n \in \mathbb{N});$$

$$\bar{A}_\omega(X) = \bigcup_{n \in \mathbb{N}} \bar{A}_n(X).$$

By $\mathcal{K}(X \times [0,1])$ we denote the hyperspace of all countable compact subsets of $X \times [0,1]$.

Theorem 2. *Let X be a separable metric space. Then:*

(1) *For $n \in \mathbb{N}$, the space $\bar{A}_n(X)$ is $F_{\sigma\delta}(\mathcal{K}(X \times [0,1]))$;*
(2) *$\bar{A}_\omega(X)$ is $F_{\sigma\delta\sigma}(\mathcal{K}(X \times [0,1]))$.*

The proof of this statement is based on some results from [3].

We then apply some characterization results of infinite-dimensional topology (see [4]) to describe the topology of spaces $\bar{I}(X)$ for noncompact locally compact separable metric spaces X.

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SKT hyperbolic and Gauduchon hyperbolic compact complex manifolds

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Definition 1. Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$, and ω be a metric on X: be a C^∞ positive definite $(1,1)$-form on X .

i) ω is *Kähler*, if $d\omega = 0$.
ii) ω is *balanced*, if $d\omega^{n-1} = 0$.
iii) ω is *Gauduchon*, if $\bar{\partial}\partial\omega^{n-1} = 0$, such a metric always exists on a compact complex manifold.

iv) ω is **SKT** (or pluriclosed), if $\partial\bar{\partial}\omega = 0$

Let $\pi_X : \widetilde{X} \longrightarrow X$ be the universal cover of X and $\widetilde{\omega} = \pi_X^\star\omega$ be the Hermitian metric on $\widetilde{X}$ that is the lift of ω. Recall that a C^∞ k-form α on X is said to be $\widetilde{d}$(bounded) with respect to ω if $\pi_X^\star\alpha = d\beta$ on $\widetilde{X}$ for some C^∞ $(k-1)$-form β on $\widetilde{X}$ that is bounded w.r.t. $\widetilde{\omega}$. (See [1] and [2]). In general, we propose the following definition which generalizes that of $\widetilde{d}$-bounded of a differential form.

Definition 2. a C^∞ k-form ϕ on X is said to be $\widetilde{(\partial+\bar{\partial})}$-bounded with respect to ω if $\pi_X^\star\phi = \partial\alpha + \bar{\partial}\beta$ on $\widetilde{X}$ for some C^∞ $(k-1)$-forms α and β on $\widetilde{X}$ that are bounded w.r.t. $\widetilde{\omega}$.

M. Gromov introduced in one of his seminal papers [1] the notion of *Kähler hyperbolicty* for a compact Kähler manifold X. The manifold X is called *Kähler hyperbolic* if X admits a Kähler metric ω whose lift $\widetilde{\omega}$ to the universal cover $\tilde{X}$ of X can be expressed as

$$\widetilde{\omega} = d\alpha$$

for a *bounded* 1-form α on $\tilde{X}$. As pointed out by Gromov, it is not hard to see that the Kähler hyperbolicity implies the Kobayashi hyperbolicity.
The Kähler hyperbolicity is generalized in [2] to what we call **balanced hyperbolicity**. This is done by replacing the Kähler metric in the Kähler hyperbolicity by a *balanced metric*. Meanwhile, a compact complex n-dimensional manifold X is said to be balanced hyperbolic if it carries a balanced metric ω such that ω^{n-1} is $\widetilde{d}$-bounded. The Brody hyperbolicity is replaced by what we call **divisorial hyperbolicity**. A compact complex manifold X is called *divisorially hyperbolic* if there exists no non-trivial holomorphic map from $\mathbb{C}^{n-1}$ to X satisfying certain *subexponential volume growth condition*.

We introduce the following

Definition 3. Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$. A Hermitian metric ω on X is said to be

(1) **SKT hyperbolic** if ω is SKT and $\widetilde{(\partial+\bar{\partial})}-$bounded with respect to ω. The manifold X is said to be SKT hyperbolic if it carries a *SKT hyperbolic metric*.
(2) **Gauduchon hyperbolic** if ω^{n-1} is $\widetilde{(\partial+\bar{\partial})}-$bounded with respect to ω. The manifold X is said to be Gauduchon hyperbolic if it carries a *Gauduchon hyperbolic metric*.

Lemma 4. *The following implication holds:*

X *is* **Kähler hyperbolic** $\Longrightarrow$ X *is* **SKT hyperbolic**

$\Downarrow$

X *is* **balanced hyperbolic** $\Longrightarrow$ X *is* **Gauduchon hyperbolic**

The following results are taken from [3]

Theorem 5. *Every* **SKT hyperbolic** *compact complex manifold is* **Kobayashi hyperbolic**.

Remark 6. An immediate observation is that, since a *SKT hyperbolic* manifold X contains no rational curves, then by Mori's cone theorem we get K_X is nef.

Theorem 7. *Every* **Gauduchon hyperbolic** *compact complex manifold is* **divisorially hyperbolic**.

Theorem 8. *Let X be a compact complex* **SKT hyperbolic** *manifold with $dim_{\mathbb{C}} X = n$. Let $\pi : \widetilde{X} \longrightarrow X$ be the universal cover of X and $\widetilde{\omega} := \pi^\star\omega$ the lift to $\widetilde{X}$ of a SKT hyperbolic metric ω on X. Fix a primitive $L^2_{\widetilde{\omega}}$-form ϕ on $\widetilde{X}$ of bidegree (p,q) with $p+q = n-1$ such that*

$$\partial\phi = 0, \qquad \bar\partial\phi = 0.$$

Then $\phi = 0$.

Corollary 9. *Let ϕ be a $(n-1,0)$-form (respectively a $(0,n-1)$-form) on a connected complete manifold $(\widetilde{X}, \widetilde{\omega})$ such that*

$$\phi \in L^2(\widetilde{X}), \qquad \partial\phi = 0, \qquad \bar\partial\phi = 0.$$

If $\widetilde{\omega} = \partial\alpha + \bar\partial\beta$ where α and β are bounded 1-forms on $\widetilde{X}$, then

$$\phi = 0.$$

Theorem 10. *Let (X, ω) be a complete Kähler manifold of dimension $2n$ and $\omega = \partial\alpha + \bar\partial\beta$ where α and β are respectively a bounded $(0,1)$ and $(1,0)$ forms on X. Then every L_2-form Ψ on X of degree $p \neq m$ satisfies the inequality*

$$\langle\psi, \Delta\psi\rangle \geq \lambda_0^2\langle\psi, \psi\rangle,$$

where λ_0 is a strictly positive constant which depends only on $n = dimX$, α and β.

Corollary 11. *Let $(\widetilde{X}, \widetilde{\omega})$ be a connected complete Kähler manifold. If $\widetilde{\omega} = \partial\alpha + \bar\partial\beta$ where α and β are bounded 1-forms on $\widetilde{X}$, then $\mathcal{H}^p_{\Delta_{\widetilde{\omega}}}(\widetilde{X}, \mathbb{C}) = 0$, unless $p = n$.*

This is a new conjecture.

Conjecture 12. *If a compact complex manifold admits a balanced hyperbolic metric and an SKT hyperbolic metric, then it admit a Kähler hyperbolic metric.*

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Invariant $*$-measures

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A triangular norm is a binary operation $*$ on $[0,1]$ which is associative, commutative, monotone (i.e. $a \le b$, $c \le d$ together imply $a * c \le b * d$), and 1 is the neutral element. In [1] the notion of $*$-measure is introduced. Given a compact Hausdorff space X, we define a $*$-measure on X as a functional $\mu\colon C(X,[0,1]) \to [0,1]$ satisfying: $\mu(c) = c$ for arbitrary $c \in [0,1]$; $\mu(\max\{\varphi,\psi\}) = \max\{\mu(\varphi),\mu(\psi)\}$ for all $\varphi,\psi\colon X \to [0,1]$; $\mu(\lambda * \varphi) = \lambda * \varphi$ for all $\lambda \in [0,1]$ and $\varphi \in C(X,[0,1])$. It is proved in [1] that the weak* topology on the set $I^*(X)$ of all $*$-measures on X makes it a compact Hausdorff space and determines a functor in the category of compact Hausdorff spaces.

Given a system of self maps $\{f_1,\dots,f_n\}$ on a compact metrizable space X and a $*$-measure on $\{1,\dots,n\}$, in a standard way one can define an analog of the Hutchinson-Barnsley operator for $*$-measures and the notion of invariant $*$-measure on X.

The talk is devoted to the question of existence of invariant $*$-measures.

Our results are in the spirit of [2] and [3] (ultrametric case). Also, in the case of metric space X, one can define a metric on the space of all $*$-measures which is a modification of a metric from [4] (in turn, the latter is a version of Bazylevych-Repovš-Zarichnyi metric [5]). This metric allows us to apply the Banach Contraction Principle to the problem of existence of invariant $*$-measures.

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Hölder Continuity of Generalized Harmonic Functions in the Unit Disc

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The main purpose of this talk is to discuss about the membership in Hölder classes for (p,q)-harmonic functions $u = K_{p,q}[f]$ such that their boundaries functions $f \in \Lambda_\beta(\mathbb{T})$.

Consider the second order partial differential operators, studied in [2], of the form

$$L_{p,q} := (1-|z|^2)\partial\overline{\partial} + pz\partial + q\overline{z}\overline{\partial} - pq, \qquad z \in \mathbb{D}, \tag{1}$$

where p, q are real parameters. We say that a function u is (p,q)-harmonic if u is twice continuously differentiable in $\mathbb{D}$ and $L_{p,q}u = 0$.

Let consider the associated Dirichlet boundary value problem of functions u, satisfying the equation $L_{p,q}u = 0$,

$$\begin{aligned} L_{p,q}u &= 0 \quad \text{in } \mathbb{D}, \\ u &= f \quad \text{on } \mathbb{T}. \end{aligned} \tag{2}$$

For $p, q \in \mathbb{R}\backslash\mathbb{Z}^-$ such that $p+q > -1$, the $(p,q)-$harmonic Poisson kernel is defined by

$$K_{p,q}(z) = c_{p,q}\,\frac{(1-|z|^2)^{p+q+1}}{(1-z)^{p+1}(1-\overline{z})^{q+1}}, \quad c_{p,q} = \frac{\Gamma(p+1)\Gamma(q+1)}{\Gamma(p+q+1)},$$

where Γ is the Gamma function.

The Solution of (2) has the following (p,q)-harmonic integral representation of $f \in L^1(\mathbb{T})$ which defined by

$$u(z) = K_{p,q}[f](z) := \frac{1}{2\pi}\int_0^{2\pi} K_{p,q}(ze^{-i\theta})f(e^{i\theta})d\theta, \quad z \in \mathbb{D}. \tag{3}$$

Remark that if $p = q = 0$, then the solution u is the classical harmonic function.

Let us recall the notion of "Hölder" continuity

Definition 1. For a bounded subset E of the complex plane, let ω be a *majorant*, i.e., a continuous increasing function on $[0,\infty)$ such that $\omega(0) = 0$ and $\omega(t)/t$ is non-increasing for $t > 0$. For a real or complex valued function f on E we write $f \in \Lambda_\omega(E)$ if there is a constant $C > 0$ such that

$$|f(z_1) - f(z_2)| \le C\omega(|z_1 - z_2|), \qquad z_1, z_2 \in E.$$

If $\omega(t) = t^\beta$, the class is simply denoted by $\Lambda_\beta(E)$ which is commonly referred to as the Hölder class for the set E of order $\beta \in (0,1]$.

The following growth estimate is useful.

Lemma 2. [3] *Let $u \in \mathcal{C}^1(\mathbb{D})$ and ω be a majorant satisfying the Dini condition, that is,*

$$\tilde{\omega}(x) := \int_0^x \frac{\omega(t)}{t}dt < \infty, \quad x > 0.$$

If f satisfies $|\partial u(z)| + |\overline{\partial} u(z)| \leq C\frac{\omega(1-|z|^2)}{1-|z|^2}$ *for all* $z \in \mathbb{D}$, *then* $u \in \Lambda_{\tilde{\omega}}(\mathbb{D})$.

As a consequence of the previous result we get the first main result

Theorem 3. [1] *Let* $p+q > 1$ *and* $0 < \beta \leq 1$. *Let* $f \in \Lambda_\beta(\mathbb{T})$ *and set* $u = K_{p,q}[f]$. *It yields*

(1) *If* $p+q \neq \beta - 1$, *then* $u \in \Lambda_{\min\{\beta, p+q+1\}}(\mathbb{D})$.

(2) *If* $p+q = \beta - 1$, *then* $u \in \Lambda_{\omega_\beta}(\mathbb{D})$, *where* $\omega_\beta(t) := t^\beta\bigg(1 - \log(t)\bigg)$.

In particular $u \in \displaystyle\bigcap_{0<\alpha<\beta} \Lambda_\alpha(\mathbb{D})$.

In particular, for $\beta = 1$, and $-1 < p+q < 0$, we have $u \in \Lambda_{p+q+1}(\mathbb{D})$ and we provide an example where $u \notin \Lambda_1(\mathbb{D})$. This example shows the failure of the stability of Lipschitz continuity in the case $p+q < 0$, i.e we provide new examples of functions $f \in \Lambda_1(\mathbb{T})$ such that $u = K_{p,q}[f] \notin \Lambda_1(\mathbb{D})$, when $p+q \in (-1, 0]$. For more details we refer the reader to [1].

Example 4. [1] Let consider two cases:

- The case $-1 < p+q < 0$: let $k \in \mathbb{Z}$,

$$u_k(z) := \frac{1}{2\pi}\int_0^{2\pi} K_{p,q}(ze^{-i\theta})e^{ik\theta}d\theta, \quad z \in \mathbb{D}.$$

- The case $p+q = 0$ and $p \neq 0$: let

$$u_0(z) := \frac{1}{2\pi}\int_0^{2\pi} K_{p,q}(ze^{-i\theta})d\theta = c_{p,q}F(-p, -q; 1; |z|^2), \quad z \in \mathbb{D}.$$

where F is the Gaussian hypergeometric function. Through these two case we can prove that one of the partial derivative of u_k (resp. u_0) is not bounded, which leads to $u_k \notin \Lambda_1(\mathbb{D})$ (resp. $u_0 \notin \Lambda_1(\mathbb{D})$).

Definition 5. A sense-preserving diffeomorphism u is said to be $K-$quasiconformal, if

$$\frac{|\partial u(z)| + |\overline{\partial} u(z)|}{|\partial u(z)| - |\overline{\partial} u(z)|} \leq K,$$

throughout the given region Ω, where $K \in [1, \infty)$ is a constant.

Under an extra condition, we preserve the same order of f and u.

Theorem 6. [1] *Let* $p+q > 1$ *and* $0 < \beta \leq 1$, $f \in \Lambda_\beta(\mathbb{T})$ *and set* $u = K_{p,q}[f]$, *be a* K*-quasiconformal mapping. Then* $u \in \Lambda_\beta(\mathbb{D})$.

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Reeb graph invariants of Morse functions, manifolds and groups

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The Reeb graph of a Morse function on a closed manifold is obtained by contracting each connected component of its level sets. There are two necessary and sufficient conditions for a finite graph to be realized as the Reeb graph of a Morse function on a given closed manifold: it needs to have the so-called good orientation and its first Betti number cannot exceed the corank of the fundamental group of the manifold. Moreover, any free quotient of this group can be represented as the Reeb epimorphism of a Morse function which is induced on fundamental groups by the quotient map from the manifold to the Reeb graph. It leads to the study of relations between the notions of equivalence of epimorphisms onto free groups, cobordism of systems of hypersurfaces and topological conjugation of Morse functions.manifold to the Reeb graph.

However, the realization of a graph as the Reeb graph of a Morse function is possible only up to a homeomorphism of graphs in general. The minimum number of degree 2 vertices in Reeb graphs of Morse functions is a strong invariant of the topology of manifold. It has three essentially different lower bounds in terms of the fundamental group, homology groups and Lusternik-Schnirelmann category. In the case of orientable 3-manifolds all of them can be improved by the inequality involving the Heegaard genus, and there is also another lower bound by a new invariant defined in terms of finite presentations of the fundamental group. We use Freiheitssatz, a fundamental fact from one-relator groups, to calculate it in some cases. The equalities in these bounds are closely related with the problem of finding a function such that the first Betti number of its Reeb graph is equal to corank. It is a one of potential geometric methods of calculating the corank, which is quite a complicated task in practise.

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Car+trailers' systems are locally nilpotentizable (a Trieste 2000 conference revisited)

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A car towing a number of passive idealized trailers is a classical kinematical model visualising so-called Goursat distributions. In the description of that series of models (indexed by the number of trailers) there are used trigonometric functions of angles between neighbouring trailers and between the car and its closest trailer. This heavily obscures the algebraic side of the models: the generated control Lie algebra is clearly not nilpotent and infinite-dimensional. Hector Sussmann asked in 1998 if the car+trailers' kinematical systems were nilpotentizable. We presented a positive answer to that question at a Trieste 2000 conference. However, recent scientific meetings show that that our result is not quite known... The aim of the talk is to make better known that result and to sketch our [old] proof of it.

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Degree theory for proper C^1 Fredholm mappings with applications to boundary value problems on the half line

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We overview elements of the definition and several properties, of a degree theory for proper C^1 Fredholm mappings of index zero [1, 2]. We establish sufficient conditions for solvability of an ODE system $\dot{v} + g(t, w) = f_1(t)$, $\dot{w} + h(t, v) = f_2(t)$ under various boundary conditions on the half line. Note that the unbounded domain prevents the use of Leray-Schauder degree. We establish sufficient conditions for solvability of a semilinear parabolic PDE $u_t - A(t)u + F(t, x, u) = f(t, x)$, once again with conditions at $t = 0$ and as $t \to \infty$. These applications illustrate methods to meet the conditions associated with the degree theory, including smoothness, properness, the Fredholm property, and the establishment of *a priori* bounds. (Note: this is an exposition of work previously published [3, 4].)

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How far apart can the projection of the centroid of a convex body and the centroid of its projection be?

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Let K be a convex body in $\mathbb{R}^n$, i.e., a compact convex set with non-empty interior. The centroid (the center of mass) of K is the point

$$c(K) = \frac{1}{|K|}\int_K x\,dx,$$

where $|K|$ denotes the volume of K and the integration is with respect to Lebesgue measure.

In this work we study the following question. Let H be a hyperplane in $\mathbb{R}^n$. Denote by $P_H c(K)$ the orthogonal projection of the centroid of K onto H and by $c(P_H K)$ the centroid of the projection of K onto H. For centrally symmetric bodies these two points coincide, but for non-symmetric bodies these points are generally different. Thus it is natural to ask how far apart these two points can be relative to some linear size of K. More precisely, we are interested in the smallest constant D_n such that for any convex body K in $\mathbb{R}^n$ we have

$$|P_H c(K) - c(P_H K)| \le D_n w_K(u),$$

where u is the unit vector parallel to the segment connecting $P_H c(K)$ and $c(P_H K)$, and $w_K(u)$ is the width of K in the direction of u, given by

$$w_K(u) = \max_{x\in K}\{\langle x,u\rangle\} - \min_{x\in K}\{\langle x,u\rangle\}.$$

Questions of this type began attracting attention several years ago in connection to Grünbaum-type inequalities for sections and projections; see [5], [2]. In particular, an analogue of the question above for sections of convex bodies is stated in [4, p. 127]. For other questions related to distances between various centroids the reader is referred to the book [1, p. 36] and the references contained therein.

Theorem 1 ([3])**.** *Let D_n, $n \geq 3$, be the smallest number such that*

$$|P_H c(K) - c(P_H K)| \leq D_n \cdot w_K(u), \tag{1}$$

for every convex body K in $\mathbb{R}^n$ and every hyperplane $H \subset \mathbb{R}^n$, where

$$u = \frac{P_H c(K) - c(P_H K)}{|P_H c(K) - c(P_H K)|},$$

provided $P_H c(K) \neq c(P_H K)$. Then

(i) $D_3 = 1 - \sqrt{\frac{2}{3}} \approx 0.1835$*; the sequence* $\{D_n\}_{n=3}^{\infty}$ *is increasing; and* $\lim_{n\to\infty} D_n \approx 0.2016$.

(ii) *Inequality (1) turns into equality if and only if K is a body obtained as follows. For a fixed hyperplane H and a vector u parallel to H, denote by θ a unit normal vector to H and take any $(n-2)$-dimensional subspace U orthogonal to u and transversal to θ. Let L_0 be any convex body in U. Denote by tL_0 the dilation of L_0 with respect to its centroid by a factor of $t = t_{max}$, which will be defined later in the proof. Let λ, μ, ν be real numbers, $\mu \neq \nu$. Define $L_1 = tL_0 + \lambda u + \mu\theta$ and $L_2 = tL_0 + \lambda u + \nu\theta$. Then K is the convex hull of L_0, L_1, and L_2. Figure 1.1 shows an example of such a body in $\mathbb{R}^3$ when $H = \{x_3 = 0\}$, $u = e_1$, and U is the linear span of e_2.*

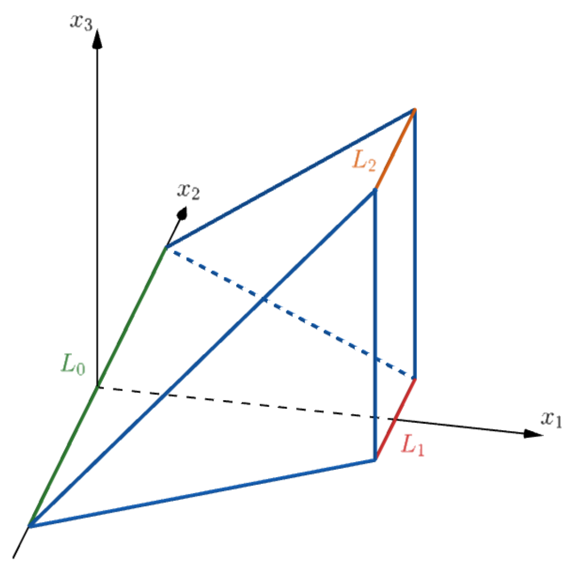

FIGURE 1.1

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Contractions of representations and realizations of Lie algebras

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Realizations (first-order differential operators) and representations (linear operators) of Lie algebras are widely applicable in modern group analysis of differential equations, in classification of gravity fields, in geometric control theory, in difference schemes for numerical solutions of differential equations, in theory of invariants, etc.

To study limit processes that connect different theories or their mathematical models it is useful to investigate contractions (limit connections) of their underling symmetries. In practice, we first study possible limit processes between abstract Lie algebras, and then, we need to find a way how to introduce similar limits in the existing realizations or representations of Lie algebras. Unfortunately, the direct application of the known contraction to a realization or representation of a Lie algebra gives several zero operators, what makes it impossible for further application to real equations.

To overcome this obstacle, we propose to construct a parameterized series of realizations and representations based on the action of the contraction matrix on the tensor of structure constants. The realizations and representations obtained in this way coincide in the limit with the corresponding realizations and representations of contracted Lie algebras. We provide the algorithm for constructing parameterized series and present a number of illustrative examples.

For clarity, let's consider main definitions. Let $\mathcal{L}_n(V)$ be the variety of n-dimensional Lie algebras (set of Lie brackets) on a vector space V over the field $\mathbb{R}$, then each n-dimensional Lie algebra $\mathfrak{g} = (V, [.,.])$ corresponds to a multiplication rule $\mu \in \mathcal{L}_n$: $\forall\, x, y \in V \quad [x, y] = \mu(x, y)$.

General linear group $\mathrm{GL}(V)$ acts on the variety of Lie brackets as follows:

$$\forall A \in \mathrm{GL}(V),\ \forall \mu \in \mathcal{L}_n \quad (A\mu)(x, y) = A^{-1}(\mu(Ax, Ay)) \quad \forall x, y \in V.$$

Consider a continuous function $U(\varepsilon) = U\colon (0, 1] \to \mathrm{GL}(V)$ and a parameterized family of Lie algebras $\mathfrak{g}_\varepsilon = (V, [.,.]_\varepsilon)$ with the Lie product defined for arbitrary elements of the vector space $[x, y]_\varepsilon = {U_\varepsilon}^{-1}[U_\varepsilon x, U_\varepsilon y]$. All such algebras are isomorphic to the initial algebra $\mathfrak{g} = (V, [.,.])$.

Definition 1. If $\forall x, y \in V$ there exists a limit

$$[x, y]_0 := \lim_{\varepsilon\to+0}[x, y]_\varepsilon = \lim_{\varepsilon\to+0} {U_\varepsilon}^{-1}[U_\varepsilon x, U_\varepsilon y]$$

then $[.,.]_0$ is a well-defined Lie bracket and Lie algebra $\mathfrak{g}_0 = (V, \mu_0)$ is called a *contraction* of the Lie algebra $\mathfrak{g}$.

Let $M \subset \mathbb{R}^m$ be an open domain. Let us denote the Lie algebra of smooth vector fields on M by $\mathrm{Vect}(M)$.

Definition 2. A *realization* of a Lie algebra $\mathfrak{g}$ in vector fields on M is a homomorphism

$$R\colon \mathfrak{g} \to \mathrm{Vect}(M).$$

Let us consider the algebra of endomorphisms $gl(V)$ of the vector space V and define a representation, which is closely related to Lie algebra module.

Definition 3. A *representation* of a Lie algebra $\mathfrak{g}$ is a homomorphism

$$\varphi\colon \mathfrak{g} \to gl(V).$$

Let us outline the algorithm in the case of realizations:

(1) Construct parameterized structure constants using the continuous function U, that do realize the desired contraction $C^{k'}_{\varepsilon,i'j'} := (U_\varepsilon)^i_{i'}(U_\varepsilon)^j_{j'}(U_\varepsilon{}^{-1})^{k'}_k C^k_{ij}$, where C^k_{ij} are structure constants of the initial Lie algebra.
(2) Calculate ε-dependent adjoint actions (using the structure constants $C^{k'}_{\varepsilon,i'j'}$), exponents and differential 1-forms: $\mathrm{ad}^\varepsilon e_i$, $\exp(-x_i \mathrm{ad}^\varepsilon e_i)$, $\omega^\varepsilon(x)$.
(3) Find the inverse transformation to obtain the vector fields $\xi^\varepsilon(x) = (\omega^\varepsilon(x))^{-1}$, that are the parameterized realization that do contracts to the realization of the contracted Lie algebra.

To conclude let us mention that contraction of the fixed realization or representation of a Lie algebra is more complicated task. Namely, in the case of realization, we first have to define it's subalgebra (studying the kernel of the linear operator in the initial point), then we have to find the equivalence transformations to the canonical realization. After that we can apply our algorithm and complete it by the inverse of the equivalence transformations.

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Geodesic orbit pseudo Riemannian nilmanifolds

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We know that in the Riemannian case, (i) for every homogeneous space, there is a reductive decomposition at the level of Lie algebras, (ii) the isometry group of a simply connected nilmanifold is the semidirect product of isometric automorphisms and translations (Wolf/Wilson), and (iii) geodesic orbit nilmanifolds are necessarily two-step nilpotent or abelian (Gordon). Neither of this is true in pseudo-Riemannian signature. However, it turns out that in low signature, some results may still be "rescued". This is a joint work (which is partially still in progress) with Joe Wolf, Zhiqi Chen and Shaoxiang Zhang.

The conditions of hypercyclicity of weighted backward shifts

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It is well known that any infinite-dimensional separable Banach space admits hypercyclic operators while finite-dimensional does not. Hypercyclicity of linear operators is a purely infinite-dimensional phenomenon. Another infinite-dimensional phenomenon is the existence of entire analytic functions of unbounded type. The weighted backward shift, introduced by Rolewicz [1], is a significant example of hypercyclic operator. On the other hand, in the talk, we will show that by using the backward shift, it is possible to construct analytic functions of unbounded type.

Definition. An analytic function f on a Banach space is said to be a function of *bounded type,* if it is bounded on all bounded subsets of X.

We denote by $H(X)$ the space of all analytic functions on X and by $H_b(X)$ the subspace of analytic functions of bounded type. It is well known that if X is infinite-dimension, then $H_b(X)$ is a proper subset of $H(X)$. Elements of $H(X) \setminus H_b(X)$ are called analytic functions of *unbounded type* [2].

Theorem. Let P_n be a sequence of n-homogeneous polynomials on a Banach space X with $\|P_n\| = 1$ and $T\colon X \to X$ a bounded linear operator satisfying

$$0 < \limsup_{n\to\infty} \|P_n \circ T^n\|^{1/n} < \infty.$$

Suppose that there exists a dense subspace $Z_0 \subset X$ such that for every $z \in Z_0$ there is a number N such that $T^N(z) = 0$. Then

$$f(x) = \sum_{n=1}^{\infty} P_n \circ T^n(x)$$

is an analytic function of unbounded type on X.

The backward shift $(x_1, x_2, \ldots,) \mapsto (x_2, x_3, \ldots,)$ in ℓ_p, $1 \le p < \infty$ or c_0 is an example of the operator T.

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Studying the properties of a superpotential using algebraic equations

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The real world as we know it occurs at energies well below the Planck scale, so it is very well described by effective field theory. These effective field theories arise as low-energy descriptions of some "vacuums" of string theory, which in some approximate schemes can be considered as solutions of the equations of motion for a compactification space. In attempts to understand the fundamentals of string theory, it has become clear that we need a better understanding of conformal theories as these are the building blocks of string vacua. Conformal theories are in general very complicated but using the renormalization group (RG) theory and the identification of fixed points of RG flow with conformal theories, we can characterize the conformal theory by the corresponding data. This approach is most powerful when applied to superconformal models with N = 2 worldsheet supersymmetry [1]. The action for a $N = 2$ supersymmetric quantum field theory takes the form

$$\int d^2zd^4\theta K(\overline{\Phi_i}, \Phi_i) + \left(\int d^2zd^2\theta W(\Phi_i) + \ c.c.\right),$$

where W is a holomorphic function of the chiral superfields Φ_i. W is not renormalized and provides us with an invariant of the renormalization group flow with which to characterize two-dimensional theories. For example, the Landau-Ginzburg super-potential $W(\Phi) = \Phi^{p+2}$ corresponds to the A-series modular invariant $N = 2$ minimal theory of level p and central charge $c = 3p/(p + 2)$. For a tensor product of minimal models we have a superpotential

$$W(\Phi_1, \ldots, \Phi_r) = \Phi_1^{p_1+2} + \ldots + \Phi_r^{p_r+2}.$$

At the fixed point of superpotential, the theory must be scale invariant, and so potential has the property that if one scales the fields according to

$$\Phi_i \to \lambda^{\omega_i}\Phi_i,$$

then the potential scales by

$$W(\lambda^{\omega_i}\Phi_i) = \lambda W(\Phi_i).$$

Such functionals are called quasi-homogeneous. The scale invariance is connected with conformal field theory. In particular for modal deformations to be considered as physical moduli of the conformal field theory, they should respect the quasi-homogeneity of the superpotential. This is a special property of $N = 2$ theories, and follows from the non-renormalization theorems. These superpotentials could be shown by checking the correspondence between the central charge c, the dimension of chiral fields, and the ring of the corresponding minimal model. This means that we can obtain the C a l a b i - Y a u manifold with the tensor product of the minimal discrete models from the point of view of L G theory [2].

We considerd different $N = 2, 3, 4$ models, calculated corresponding central charges, $c = 3, 6, 9$ and investigated the forms and roots of such manifolds for singular 2-fold, or K3 surface, defined by the following polynomials [3]

$$F_{A_{N-1}} = x_1^N + x_2^2 + x_3^2, (N \geq 2)$$

$$F_{E_6} = x_1^4 + x_2^3 + x_3^2$$
$$F_{E_8} = x_1^5 + x_2^3 + x_3^2$$

Es example, for polynomial of the form

$$5x^5 + 6y^2 + 3z^2 = 0 \tag{1}$$

we have the following surface and roots in complex plane

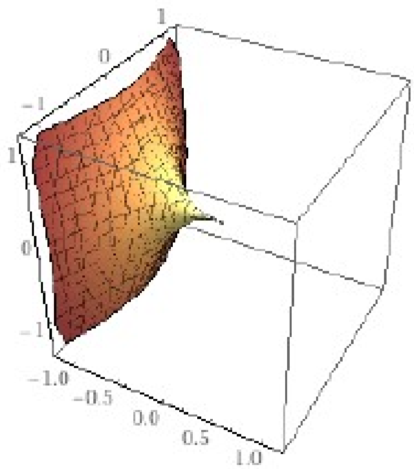

FIGURE 0.1. Surface of the equation (1).

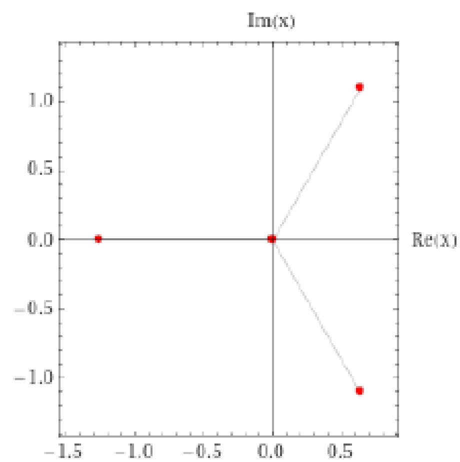

FIGURE 0.2. Roots of equation (1).

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On critical submanifolds of the Willmore energy in four dimensions

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We establish a rigidity result for the critical points, with boundary, of the four dimensional Willmore energy (see [13] where this energy was studied from analytical standpoint). These critical points satisfy a 4-Willmore equation which is a sixth order nonlinear elliptic partial differential equation. We establish several curvature estimates and prove that four dimensional Willmore submanifold with totally geodesic boundary condition are umbilic.

The rigidity of several kinds of submanifolds has been widely studied in literature under different contexts. For instance, while some rigidity results for manifolds with bounded Ricci curvature were obtained in [2] other studies have focused on minimal submanifolds [3, 5, 6, 7, 11, 14, 16], critical points of the Willmore functional [8, 9, 10] and hypersurfaces of constant weighted mean curvature [1, 4, 15]. In [12], McCoy and Wheeler considered surfaces Σ immersed into $\mathbb{R}^3$ which are critical points of the functional

$$\int_\Sigma |\nabla H|^2 \, d\mu$$

and whose second fundamental form satisfies the smallness condition

$$\int_\Sigma |h|^2 \, d\mu \le \varepsilon$$

where ε is a small universal constant. They obtained the following result.

Theorem 1. *Let $f : \Sigma \to \mathbb{R}^3$ be an immersion satisfying*

$$\Delta^2 H + |h|^2 \Delta H - (h_0)^{ij} \nabla_i H \nabla_j H = 0$$

with the boundary conditions

$$|h| = 0 \quad \textit{and} \quad \nabla_\eta H = \nabla_\eta \Delta H = 0.$$

If f also satisfies $\int_\Sigma |h|^2 d\mu \le \varepsilon$ for some sufficiently small $\varepsilon > 0$, then the immersed surface $f(\Sigma)$ is part of a flat plane, where η is the unit conormal to the boundary of Σ.

Our main result is the following rigidity theorem for critical points of the energy $\mathcal{E}(\Sigma)$.

Theorem 2. *Let $\vec{\Phi} : \Sigma \to \mathbb{R}^m$ be an immersion of a 4-dimensional manifold Σ satisfying $\int_\Sigma |\vec{h}|^2 d\mu \le \varepsilon$ and $\int_\Sigma |\vec{h}|^4 d\mu \le \varepsilon$ for some sufficiently small $\varepsilon > 0$. If $\vec{\Phi}$ also satisfies*

$$\vec{\mathcal{W}} = \vec{0} \tag{1}$$

together with the boundary conditions

$$\pi_{\vec{\eta}} \nabla \Delta_\perp \vec{H} = \pi_{\vec{\eta}} \nabla \vec{H} = \vec{0} \quad \textit{and} \quad \vec{h} = \vec{0} \tag{2}$$

where

$$\begin{aligned}\vec{\mathcal{W}} := &-\frac{1}{2}\Delta_\perp^2\vec{H} - \frac{1}{2}(\vec{h}_{ik}\cdot\Delta_\perp\vec{H})\vec{h}^{ik} - 4|\pi_{\vec{n}}\nabla\vec{H}|^2\vec{H} + 2\pi_{\vec{n}}\nabla_j((\vec{h}_i^j\cdot\nabla^i\vec{H})\vec{H}) - 2\pi_{\vec{n}}\nabla_j((\vec{H}\cdot\vec{h}_i^j)\pi_{\vec{n}}\nabla^i\vec{H})\\ &+ 2(\pi_{\vec{n}}\nabla_i\vec{H}\cdot\pi_{\vec{n}}\nabla_j\vec{H})\vec{h}^{ij} - \frac{1}{2}\Delta_\perp((\vec{H}\cdot\vec{h}^{ij})\vec{h}_{ij}) - 2\pi_{\vec{n}}\nabla_i\nabla_k((\vec{H}\cdot\vec{h}^{ik})\vec{H}) - 28|\vec{H}|^4\vec{H}\\ &- \frac{1}{2}(\vec{H}\cdot\vec{h}^{ij})(\vec{h}_{ij}\cdot\vec{h}_{pq})\vec{h}^{pq} - 4(\vec{H}\cdot\vec{h}_{ij})(\vec{H}\cdot\vec{h}_k^i)\vec{h}^{jk} + 4|\vec{H}\cdot\vec{h}|^2\vec{H} + 7\Delta_\perp(|\vec{H}|^2\vec{H}) + 7|\vec{H}|^2(\vec{H}\cdot\vec{h}_{ij})\vec{h}^{ij}\end{aligned}$$

then the submanifold Σ *is umbilic with totally geodesic boundary.*

Note that the Willmore equation (1) is a sixth order nonlinear partial differential equation.

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Fermat–Torricelli sets of finite sets of points in Euclidean plane

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Definition 1 ([8])**.** Let (X, ρ) be a metric space and $x_1, \ldots, x_n \in X$ be a finite collection of points in X. A point $\overline{x} \in X$ is called a *Fermat–Torricelli point* for $x_1, \ldots, x_n$ whenever for each $x \in X$ the following inequality holds true:

$$\sum_{k=1}^{n} \rho(\overline{x}, x_k) \leq \sum_{k=1}^{n} \rho(x, x_k).$$

Definition 2. *Fermat–Torricelli set* for fixed points $\{x_1, \ldots, x_n\}$ is a set of all Fermat–Torricelli points for this collection of points.

In the case when $X = \mathbb{R}^n$ is the Euclidean space with the standard metric, then for every finite collection of points $x_1, \ldots, x_n \in \mathbb{R}^n$ the set of its Fermat–Torricelli points is non-empty, convex and compact. The problem of finding the Fermat–Torricelli set is called the *Fermat–Torricelli problem.*

This problem has both geometric and probabilistic interpretation. We can describe discrete probabilistic space $\Omega = \{x_1, \ldots, x_n\}$ with a probabilistic measure P on it, so that $\forall k \in \{1, \ldots, n\} : P(x_k) = \dfrac{1}{n}$. If for any $x_0 \in (X, \rho)$ we define a random variable $\xi_{x_0}(x) := \rho(x, x_0), x \in \Omega$, then Fermat–Torricelli set is the set of those $x_0 \in (X, \rho)$, for which random variable ξ_{x_0} has the least mathematical expectation.

Theorem 3. *Let A be the Fermat–Torricelli set for a collection $\{x_1, \ldots, x_n\}$, and B be the Fermat–Torricelli set for a collection $\{y_1, \ldots, y_k\}$ in Euclidean metric space $(\mathbb{R}^m, \rho)$ with standard metric. Assume that all points $x_1, \ldots, x_n, y_1, \ldots, y_k$ are mutually distinct. Then if $A \bigcap B \neq \varnothing$, then $A \bigcap B$ is the Fermat–Torricelli set for $\{x_1, \ldots, x_n, y_1, \ldots, y_k\}$.*

Since now we will name Euclidean metric space $(\mathbb{R}^2, \rho)$ with standard metric merely Euclidean plane.

Corollary 4. *If mutually distinct points x_1, x_2, x_3, x_4 in the Euclidean plane are vertices of a convex quadrilateral, then the point of intersection of its diagonals is a unique Fermat–Torricelli point for x_1, x_2, x_3, x_4.*

Corollary 5. *Let x_1, x_2, x_3, x_4 be mutually distinct points in the Euclidean plane laying on the same line in the given order. Then Fermat–Torricelli set of these points is the following set*

$$A = \{\alpha x_2 + (1 - \alpha)x_3 \,|\, \alpha \in [0; 1]\}.$$

Corollary 6. *Let x_1, x_2, x_3 be the vertices of some triangle in the Euclidean plane, and x_4 be some other point which lays on the side of triangle between x_2 and x_3. Then x_4 is a unique Fermat–Torricelli point of x_1, x_2, x_3, x_4.*

Corollary 7. *Let $A_1A_2\ldots A_n$ be a regular polygon with an even number of vertices. Then its center of gravity is a unique Fermat–Torricelli point of its vertices.*

Theorem 8. *Let $x_1,\ldots,x_n, n\geq 3$ be distinct points in the Euclidean plane. Then the following statements hold.*

1) *If $x_1,\ldots,x_n$ lay on the same line in the given order and n is any even number, then the set*

$$A := \{x \in X \mid x = \alpha x_{\frac{n}{2}} + (1-\alpha)x_{\frac{n}{2}+1}, \alpha \in [0;1]\}$$

is a Fermat–Torricelli set for them.

2) *If $x_1,\ldots,x_n$ lay on the same line in the given order and n is an odd number, then the point $x_{\frac{n-1}{2}+1}$ is a Fermat–Torricelli set for them.*
3) *If $x_1,\ldots,x_n$ do not lay on the same line, then their Fermat–Torricelli point is unique.*

Lemma 9. *There is a unique point inside of triangle from which every side of triangle is visible under angle $120°$ if and only if every angle of this triangle is less than $120°$.*

Different sources name this point in different ways: *Fermat point*, *Torricelli point*, and even *Steiner point* [8]. We will define it as *Steiner point for respective triangle.*

Theorem 10 ([8])**.** *Let x_1,x_2,x_3 be vertices of triangle in Euclidean plane every angle of which is less than $120°$. Then the Steiner point for this triangle is a unique Fermat–Torricelli point for x_1,x_2,x_3.*

Theorem 11 ([8])**.** *Let x_1,x_2,x_3 be vertices of triangle in Euclidean plane one of whose angles is not less than $120°$. Then the vertice, whose angle of triangle is not less than $120°$, is a unique Fermat–Torricelli point of its vertices.*

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Degenerations of complex associative algebras of dimension three

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Let $\mathbf{\Lambda}_3(\mathbb{C})\,(=\mathbb{C}^{27})$ be the space of structure vectors of 3-dimensional algebras over $\mathbb{C}$ considered as a G-module via the action of $G = \mathrm{GL}(3,\mathbb{C})$ on $\mathbf{\Lambda}_3(\mathbb{C})$ 'by change of basis'. We determine the complete degeneration picture inside the algebraic subset $\mathcal{A}_3^s$ of $\mathbf{\Lambda}_3(\mathbb{C})$ consisting of associative algebra structures via the corresponding information on the algebraic subsets $\mathcal{L}_3$ and $\mathcal{J}_3$ of $\mathbf{\Lambda}_3(\mathbb{C})$ of Lie and Jordan algebra structures respectively. This is achieved with the help of certain G-module endomorphisms ϕ_1, ϕ_2 of $\mathbf{\Lambda}_3(\mathbb{C})$ which map $\mathcal{A}_3^s$ onto algebraic subsets of $\mathcal{L}_3$ and $\mathcal{J}_3$ respectively.

This is a joint work with Nataliya M. Ivanova

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Several forms of the geometric Lusternik-Schnirel'mann category

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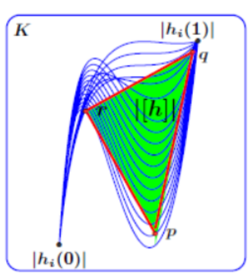

Fig. 1.1 → Fig. 1.2

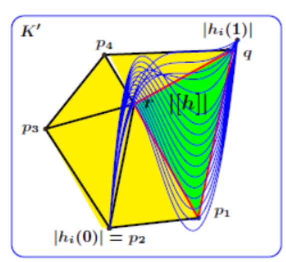

FIGURE 0.1. Fig. 1.1 from Theorem 1.6 & Fig. 1.2 from Corollary 1.8.

This paper introduces results for several forms of the geometric Lusternik-Schirel'mann categories (LS gcat).

1. Geometric Lusternik-Schnirel'mann Category

Let $h : I \to K$ be a continuous map called **homotopy** (briefly, **path** in a space K. A **homotopic class** for different maps h (denoted by $[h] = \left\{h_0, \dots, h_i, \dots, h_{n-1[n]}\right\}$ with $[n] = \text{mod } n \in \mathbb{Z}^+$) is a collection of $h_{i[n]}$ homotopic maps that have the same endpoints, namely, $h_i(0)$ and $h_i(1)$. The geometric realization of $[n]$ (denoted by $|[h]|$) is a collection of sinusoidal curves, each being the geometric realization of a path h.

Definition 1.1 (Geometric LS Category). [1] For a topological space X, the geometric category of X is the minimal covering of X with contractible open subsets of X.

Lemma 1.2. *Let $\boldsymbol{h}$ be a homotopic sinusoidal path. The geometric realization $|\boldsymbol{h}|$ is a planar sinusoidal curve.*

Lemma 1.3. *Let $[\boldsymbol{h}]$ be a collection of homotopic paths with common endpoints. The geometric realization $|[\boldsymbol{h}]|$ is a collection space filling planar curves.*

Lemma 1.4. *Let $\triangle \boldsymbol{pqr}$ be a planar filled triangle in a space K, geometric realization $|[\boldsymbol{h}]|$ such that each path h has endpoints $h(0), h(1) \in \boldsymbol{K} \setminus \triangle \boldsymbol{pqr}$. Then $\underset{i\to\infty}{limit}\, h_i \in [h] \supseteq \triangle \boldsymbol{pqr}$.*

Lemma 1.5. *Let h be a homotopic path in space K.*

- 1^o *Every path $\boldsymbol{h}$ is contractible.*
- 2^o *There exists a minimal $|[\boldsymbol{h}]|$ in space K with $h_i \in [h]$ with the same boundary endpoints $\boldsymbol{h(0), h(1)} \in \boldsymbol{\partial \triangle pqr}$ such that $|[\boldsymbol{h}]|$ covers triangle $\triangle \boldsymbol{pqr} \subset K$.*
- 3^o *Every planar triangle in K has a minimal covering $|[\boldsymbol{h}]|$.*

Theorem 1.6. *There exists $\boldsymbol{gcat}(|[\boldsymbol{h}]| \in \boldsymbol{2^K})$ such that $\boldsymbol{min}\, |[\boldsymbol{h}]| \supseteq \triangle \boldsymbol{pqr} \in \boldsymbol{2^K}$.*

Example 1.7. From Theorem 1.6, the triangle $\triangle \boldsymbol{pqr}$ in Fig. 0.1 has a $|[\boldsymbol{h}]|$ minimal covering, which is a $gcat(T)$.

A **cluster of triangles** $\boldsymbol{\{\triangle pqr\}}$ in a Euclidean space K is a collection of triangles attached to a common vertex. From Theorem 1.6, we have

Corollary 1.8. *Let $|\{[\boldsymbol{h}]\}| \subset \boldsymbol{K}$ such that each $\boldsymbol{h} \in \{[\boldsymbol{h}]\} \subset \boldsymbol{K} \setminus \{\triangle \boldsymbol{pqr}\}$ has the same endpoints. Then $\exists \boldsymbol{gcat}(|\{[\boldsymbol{h}]\}|)$: $\boldsymbol{min}\, |\{[\boldsymbol{h}]\}| \supseteq \{\triangle \boldsymbol{pqr}\} \in \boldsymbol{2^K}$.*

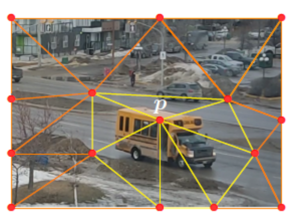

Figure 1.2. Delaunay Triangle cluster minimally covering bus in video frame foreground.

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Example 1.9. From Corollary 1.8, there is a $\boldsymbol{gcat}(|\{[\boldsymbol{h}]\}|$ such that $\boldsymbol{gcat}(|\{[\boldsymbol{h}]\}|$ is a minimal covering of the triangle cluster $\boldsymbol{\{\triangle pqr\}} \cap \boldsymbol{p}$, which is a bounded region in Fig. 0.1.

2. Minimal Video Frame Foreground Object Covering

Delaunay triangulations[2] represent pieces of a continuous space in form of triangles with edges attached to selected vertices[3].

Theorem 2.1. *Let $\mathcal{TC}$ be a Delaunay triangle cluster with k triangles minimally covering a planar bounded region $E \in 2^{\mathbb{R}^2}$. Then $\boldsymbol{gcat(\mathcal{TC}) = k}$.*

Example 2.2. With restrictions on the selection of vertices (e.g., centroids), we obtain a minimal cluster $\mathcal{TC}$ of k triangles covering a bus, which is a bounded region in a Delauany triangulation of the video frame foreground in Fig. 1.2. Hence, from Theorem 2.1, $\boldsymbol{gcat(\mathcal{TC}) = k}$.

Fixed point theorem for mappings contracting perimeters of triangles and its generalizations

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We establish two generalizations of the fixed point theorem for mappings contracting perimeters of triangles. In the first case we consider these mappings in semimetric spaces with triangle functions introduced by M. Bessenyei and Z. Páles. Such approach allows us to obtain corollaries for different types of semimetric spaces. In the second case we establish the fixed point theorem in ordinary metric spaces for more general class of mappings than mappings contractive perimeters of triangles.

Let X be a nonempty set. Recall that a mapping $d\colon X\times X \to \mathbb{R}^+$, $\mathbb{R}^+ = [0,\infty)$ is a *metric* if for all $x, y, z \in X$ the following axioms hold: (i) $(d(x, y) = 0) \Leftrightarrow (x = y)$; (ii) $d(x, y) = d(y, x)$; (iii) $d(x, y) \leqslant d(x, z) + d(z, y)$. The pair (X, d) is called a *metric space*. If only axioms (i) and (ii) hold then d is called a *semimetric*. A pair (X, d), where d is a semimetric on X, is called a *semimetric space*.

In 2017 M. Bessenyei and Z. Páles [1] introduced a definition of a triangle function $\Phi\colon \overline{\mathbb{R}}_+^2 \to \overline{\mathbb{R}}^+$ for a semimetric d. We use this definition in a slightly different form restricting the domain and the range of Φ by $\mathbb{R}_+^2$ and $\mathbb{R}^+$, respectively.

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Definition 1. Consider a semimetric space (X, d). We say that $\Phi\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is a *triangle function* for d if Φ is symmetric and monotone increasing in both of its arguments, satisfies $\Phi(0,0) = 0$ and, for all $x, y, z \in X$, the generalized triangle inequality

$$d(x, y) \leqslant \Phi(d(x, z), d(z, y))$$

holds.

Definition 2. Let (X, d) be a semimetric space with $|X| \geqslant 3$. We shall say that $T\colon X \to X$ is a *mapping contracting perimeters of triangles* on X if there exists $\alpha \in [0, 1)$ such that the inequality

$$d(Tx, Ty) + d(Ty, Tz) + d(Tx, Tz) \leqslant \alpha(d(x, y) + d(y, z) + d(x, z)) \tag{1}$$

holds for all three pairwise distinct points $x, y, z \in X$.

Note that the requirement for $x, y, z \in X$ to be pairwise distinct is essential. One can see that otherwise this definition is equivalent to the definition of contraction mapping.

Theorem 3. *Let (X, d), $|X| \geqslant 3$, be a complete semimetric space with the triangle function Φ satisfying the following three conditions:*

1) The inequality

$$\Phi(k\xi, k\eta) \leqslant k\Phi(\xi, \eta)$$

holds for all $k, \xi, \eta \in \mathbb{R}^+$.

2) For every $0 \leqslant \alpha < 1$ there exists $C(\alpha) > 0$ such that for every $p \in \mathbb{N}^+$ the inequality

$$\Phi(1, \Phi(\alpha, \Phi(\alpha^2, ..., \Phi(\alpha^{p-1}, \alpha^p)))) \leqslant C(\alpha)$$

holds.

3) Φ is continuous at $(0, 0)$.

Let the mapping $T\colon X \to X$ satisfy the following two conditions:

(i) $T(T(x)) \neq x$ for all $x \in X$ such that $Tx \neq x$.

(ii) T is a mapping contracting perimeters of triangles on X.

Then T has a fixed point. The number of fixed points is at most two.

Corollary 4. *Theorem 3 holds for semimetric spaces with power triangle functions $\Phi(x, y) = (x^q + y^q)^{\frac{1}{q}}$ if $q > 0$.*

If the usual triangle inequality is replaced by $d(x, y) \leqslant K(d(x, z) + d(z, y))$, $K \geqslant 1$, then (X, d) is called a *b-metric space*. The definition of a b-metric space was introduced by Czerwik [2].

Corollary 5. *Theorem 3 holds for b-metric spaces if $\alpha K < 1$, where α is the coefficient in (1).*

Definition 6. Let (X, d) be a metric space with $|X| \geqslant 3$ and let functions $F, G\colon \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ be such that for all $\xi, \eta, \zeta \in \mathbb{R}^+$ the following conditions hold:

$$F(\eta, \xi, \zeta) = F(\xi, \eta, \zeta) = F(\xi, \zeta, \eta),$$
$$G(\eta, \xi, \zeta) = G(\xi, \eta, \zeta) = G(\xi, \zeta, \eta),$$
$$G(\xi, \eta, \zeta) \geqslant \xi,$$

$$F(\xi,\eta,\zeta) \geqslant G(\xi,\eta,\zeta),$$
$$G(0,0,0) = 0 \text{ and } G \text{ is continuous at } (0,0,0).$$

We shall say that $T\colon X \to X$ is an (F,G)-*contracting mapping* on X if there exists $\alpha \in [0,1)$ such that the inequality

$$F(d(Tx,Ty),d(Ty,Tz),d(Tx,Tz)) \leqslant \alpha\, G(d(x,y),d(y,z),d(x,z))$$

holds for all three pairwise distinct points $x,y,z \in X$.

Theorem 7. *Let (X,d), $|X| \geqslant 3$, be a complete metric space and let $T\colon X \to X$ be a mapping satisfying the following two conditions:*

(i) $T(T(x)) \neq x$ for all $x \in X$ such that $Tx \neq x$.

(ii) T is an (F,G)-contracting mapping on X.

Then T has a fixed point. The number of fixed points is at most two.

If in Theorem 3 we set $\Phi(x,y) = x + y$ or in Theorem 7 we set $F(\xi,\eta,\zeta) = G(\xi,\eta,\zeta) = \xi + \eta + \zeta$, then we get the following.

Corollary 8. *Let (X,d), $|X| \geqslant 3$, be a complete metric space and let the mapping $T\colon X \to X$ satisfy the following two conditions:*

(i) $T(T(x)) \neq x$ for all $x \in X$ such that $Tx \neq x$.

(ii) T is a mapping contracting perimeters of triangles on X.

Then T has a fixed point. The number of fixed points is at most two.

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Structure of codimensional one flows on the 2-sphere with holes

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First, we consider gradient vector fields on a sphere. Since the function increases along each trajectory, the field has no cycles and polycycles. In general position, a typical gradient field is a Morse field (Morse-Smale field without closed trajectories). In typical one-parameter families of gradient vector fields, two types of bifurcations are possible: saddle-node and saddle connection. The corresponding vector fields at the time of the bifurcation are fields of codimension one. In our case, they completely determine the topological type of the bifurcation. To classify Morse fields, a cell complex (diagram) is often used, in which cells of dimension n are stable manifolds of singular points with Morse index equal to n. We apply this approach to the classification of vector fields of codimension one.

Without loss of generality, we assume that under bifurcation (as the parameter increases), the number of singular points does not increase. The saddle-node bifurcation is defined by a pair of cells corresponding to the singular points participating in the bifurcation. We mark this pair on the diagram with a green arrow or a triangle. A saddle-node bifurcation in the diagram corresponds to a point of degree 3, where two edges (half-edges) are opposite and the third is perpendicular to them.

Then, the separatrix that connects the saddle with the node (source or sink) contracts to a point under the saddle-node bifurcation.

We describe all possible structures of Morse flows on S^2 with holes using separatrix diagrams and methods of papers [1, 2, 3, 4, 5].

Theorem 1. [6, 7] *The following types of gradient bifurcations are possible on spheres with holes:*

SN – internal saddle node; SC – internal saddle connection; BSN - boundary saddle node; BDS – boundary double saddle; HN – semi-boundary saddle node (node); HS – semi-boundary saddle-node (saddle); HSC – semi-boundary saddle connection; BSC – saddle connection of points on the boundary.

All possible structures of Morse flows and typical one-parameter bifurcations on spheres with holes in which no more than six singular points are given in Table 1.

Number of points	Morse	SN	SC	BSN	BDS	HN	HS	HSC	BSC
3 on D^2	2	0	0	0	0	2	0	0	0
4 on D^2	5	2	0	2	0	0	2	4	0
5 on D^2	7	8	0	2	0	6	8	4	0
6 on D^2	22	30	7	22	5	12	38	6	2
4 on $S^1 \times I$	2	0	0	0	0	0	0	0	1
5 on $S^1 \times I$	4	0	0	0	10	0	0	2	2
6 on $S^1 \times I$	14	4	2	14	6	4	18	10	9
6 on $F_{0,3}$	2	0	0	0	0	0	0	0	4

TABLE 1.1. Number of Morse flows and bifurcations on S^2 with holes (number of points befor bifurcation)

In what follows, we consider arbitrary, possibly non-gradient, flows on D^2. The optimal flow is the flow that has the least number of singular points among the flows of its type.

Theorem 2. *On a two-dimensional disk, there exist the following optimal codimensional one flow structures with degenerate singularities in the interior:*

SN: with a saddle knot - two (opposite);
HC: with a homoclonic cycle – two;
AN: Andronov-Hopf – two;
SL: with a saddle loop – two;
PC: with a parabolic cycle - two;
SC: with saddle ligament – six.

With singularities on the boundary, there exist the following optimal flows:

BSN: boundary saddle knot - two;
BHC: boundary saddle knot with a homoclinic boundary – two;
BDS: boundary double saddle – two;
BDSH: boundary double saddle with homoclinic boundary – one;

HN: semi-boundary saddle node (node) - two;
HS: semi-boundary saddle node (saddle) – four;
BDN: double nod on the boundary – two;
BDNH: double node with a homoclinic boundary – two;
HSC: semi-boundary saddle connection – two;
BSC: a connection of saddles on the boundary – three.
If the boundary is a parabolic cycle:
BPC: boundary parabolic cycle – two flow structures.

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Convex bodies of constant width with exponential illumination number

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Borsuk's number $f(n)$ is the smallest integer such that any set of diameter 1 in the n-dimensional Euclidean space can be covered by $f(n)$ sets of smaller diameter. Currently best known asymptotic upper bound $f(n) \leq (\sqrt{3/2}+o(1))^n$ was obtained by Shramm (1988) and by Bourgain and Lindenstrauss (1989) using different approaches. Bourgain and Lindenstrauss estimated the minimal number $g(n)$ of open balls of diameter 1 needed to cover a set of diameter 1 and showed $1.0645^n \leq g(n) \leq (\sqrt{3/2}+o(1))^n$. On the other hand, Schramm used the connection $f(n) \leq h(n)$, where $h(n)$ is the illumination number of n-dimensional convex bodies of constant width, and showed $h(n) \leq (\sqrt{3/2} + o(1))^n$. The best known asymptotic lower bound on $h(n)$ is subexponential and is the same as for $f(n)$, namely $h(n) \geq f(n) \geq 1.2255^{\sqrt{n}}$ for large n established by Raigorodskii (1999). In 2015 Kalai asked if an exponential lower bound on $h(n)$ can be proved.

We show $h(n) \geq (\cos(\pi/14) + o(1))^{-n}$ by constructing the corresponding n-dimensional bodies of constant width, which answers Kalai's question in the affirmative. The construction is based on a geometric argument combined with a probabilistic lemma establishing the existence of a suitable covering of the unit sphere by equal spherical caps having sufficiently separated centers. The lemma also allows to improve the lower bound of Bourgain and Lindenstrauss to $g(n) \geq (2/\sqrt{3} + o(1))^n \approx 1.1547^n$.

Bifurcation points in random dynamical systems

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Let (M, ρ) be a locally compact separable metric space. By a continuous flows of mappings of M we will understand a family $(\theta_{s,t})_{-\infty<s\leq t<\infty}$, such that

- for all $s \leq t$ $\theta_{s,t} : M \to M$;
- for all $(s, x) \in \mathbb{R} \times M$ the mapping $t \mapsto \theta_{s,t}(x)$ is continuous and satisfies $\theta_{s,s}(x) = x$;
- for all $r \leq s \leq t$ $\theta_{s,t} \circ \theta_{r,s} = \theta_{r,t}$.

If $(\theta_{s,t})_{-\infty<s\leq t<\infty}$ is a continuous flow of mappings of M and $\mathcal{D} = \{(s_n, x_n) : n \geq 1\}$ is a countable dense set in $\mathbb{R} \times M$, then one can consider a sequence of continuous functions $\Phi_n(t) = \theta_{s_n,t}(x_n)$, $t \in [s_n, \infty)$, with the property

$$\max(s_n, s_m) \leq s, \Phi_n(s) = \Phi_m(s) \Rightarrow \Phi_n|_{[s,\infty)} = \Phi_m|_{[s,\infty)} \qquad (1).$$

We are interested in the problem of recovering the flow $(\theta_{s,t})_{-\infty<s\leq t<\infty}$ from the sequence of continuous functions $(\Phi_n)_{n\geq 1}$, $\Phi_n \in C([s_n, \infty), M)$, that satisfy (1). Such problem naturally arises in the theory of stochastic flows. For example, if $\theta_{s,\cdot}(x)$ denotes the solution of the stochastic differential equation

$$dX(t) = a(X(t))dt + b(X(t))dW(t), \ X(s) = x, \qquad (2)$$

where W is a Brownian motion and a and b are continuously differentiable functions bounded together with their derivatives, then for all $r \leq s \leq t$ and $x \in M$, $\theta_{s,t}(\theta_{r,s}(x)) = \theta_{r,t}(x)$ almost surely. However, the equality $\theta_{s,t} \circ \theta_{r,s} = \theta_{r,t}$ may not hold simultaneously for all $r \leq s \leq t$. This fact limits the possibility to apply the dynamic systems technique to the study of stochastic flows. The usual way to deal with this issue is to consider solutions of (2) for some dense sequence of initial values (s_n, x_n) and define solutions for other initial values by a limiting procedure. This strategy works well for stochastic flows of solutions to stochastic differential equations with sufficiently regular coefficients [1]. However, a lot of important stochastic flows are either generated by singular stochastic differential equations, or are not generated by stochastic differential equations at all [2]. This motivates the general question of a possibility to extend a sequence of continuous mappings $(\Phi_n)_{n\geq 1}$ that satisfies

(1) to a continuous flow $(\theta_{s,t})_{-\infty<s\le t<\infty}$ of mappings of M in the sense that $\Phi_n(t) = \theta_{s,t}(\Phi_n(s))$, $s_n \le s \le t$.

Our main result is the following. Assume that $(\Phi_n)_{n\ge1}$ is a sequence of continuous mappings, $\Phi_n \in C([s_n,\infty), M)$, that satisfies (1) and is such that the sequence $((s_n, \Phi_n(s_n)))_{n\ge1}$ is dense in $\mathbb{R}\times M$, and for every compact $L \subset \mathbb{R}\times M$ the set

$$\left\{\Phi_n|_{[s,\infty)} : s_n \le s, (s, \Phi_n(s)) \in L\right\}$$

is relatively compact with respect to the topology of uniform convergence on bounded intervarls. Consider sets $\mathcal{K}_x^{s,t} = \bigcap_{\varepsilon>0} \overline{\{\Phi_n|_{[s,t]} : s_n \le s, \rho(\Phi_n(s), x) \le \varepsilon\}}$, and let

$$E = \{(s,x) \in \mathbb{R}\times M : \forall t > s\ \mathcal{K}_x^{s,t} \text{ contains at least two distinct functions}\}.$$

Assume that F is a closed subset of $\mathbb{R}\times M$, such that $E \subset F$.

Theorem 1. *Let $(\theta_{s,t} : -\infty < s \le t < \infty)$ be a family of mappings of M. Define*

$$\sigma_x^s = \inf\{t > s : \theta_{s,t}(x) \in F\}.$$

Assume that

- *for all $t \in (s, \sigma_x^s)$, $\theta_{s,t}(x) \in \mathcal{K}_x^{s,t}$;*
- *if $s_n \le s \le t$, then $\theta_{s,t}(\Phi_n(s)) = \Phi(t)$;*
- *if $\sigma_x^s \le t$, then $\theta_{\sigma_x^s,t}(\theta_{s,\sigma_x^s}) = \theta_{s,t}(x)$;*
- *if $t > \sigma_x^s$, and $\theta_{s,t}(x) \in F$, then there exists $n \ge 1$, such that $s_n \le t$ and $\theta_{s,\cdot}(x)|_{[t,\infty)} = \Phi_n|_{[t,\infty)}$.*

Then for all $r \le s \le t$ $\theta_{s,t} \circ \theta_{r,s} = \theta_{r,t}$.

We will give applications of the theorem to analogues of Arratia and Burdzy-Kaspi flows on metric graphs.

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On symmetries of sections of convex bodies

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Abstract: Christos Saroglou and Sergii Myroshnychenko proved [1] that a convex origin-symmetric body in $\mathbb{R}^n$, $n \ge 3$, with central sections having symmetries of a cube, must be a Euclidean ball. We will discuss several results on floating bodies related to this problem.

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Fuzzy metrization of spaces of $\star$-measures

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In [1], a fuzzy metrization of spaces if idempotent measures is constructed. The idempotent measures are known to be counterparts of the probability measures in the idempotent mathematics (see [4] for detailed exposition of topological aspects of idempotent measures).

Definition 1. A binary operation $*\colon [0,1]\times[0,1]\to[0,1]$ is a *continuous t-norm* if it satisfies the following conditions.

(1) $*$ is associative and commutative,
(2) $*$ is continuous,
(3) $a*1=a$ for all $a\in[0,1]$,
(4) $a*b\le c*d$ whenever $a\le c$ and $b\le d$, for each $a,b,c,d\in[0,1]$.

Definition 2. A 3-tuple $(X,M,*)$ is said to be a *fuzzy metric space* [2] if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set on $X\times X\times(0,\infty)$ satisfying the following conditions for all $x,y,z\in X$ and $s,t\in(0,\infty)$:

(1) $M(x,y,t)>0$;
(2) $M(x,y,t)=1$ if and only if $x=y$;
(3) $M(x,y,t)=M(y,x,t)$;
(4) $M(x,y,t)*M(y,z,s)\le M(x,z,t+s)$;
(5) the function $M(x,y,\cdot)\colon(0,\infty)\to[0,1]$ is continuous.

If $(X,M,*)$ is a fuzzy metric space, $(M,*)$ will be called a fuzzy metric on X.

Let $\star$ be a triangular norm. A functional $\mu\colon C(X,[0,1])\to[0,1]$ is said to be an $\star$-measure if the following holds (c_X is the constant function with value c):

(1) $\mu(c_X)=c$ for all $c\in[0,1]$;
(2) $\mu(\varphi\oplus\psi)=\mu(\varphi)\oplus\mu(\psi)$;
(3) $\mu(\lambda\star\varphi)=\lambda\star\mu(\varphi)$.

(Here, $\oplus$ denotes the max operation.)

The spaces $I^\star(X)$ of $\star$-measures on compact Hausdorff spaces X are endowed with the weak* topology [3].

The aim of the talk is to provide a fuzzy metrization of the spaces $I^\star(X)$ on fuzzy metric spaces $(X,M,*)$. To this end, we identify the spaces $I^\star(X)$ with subsets of the hyperspace of $X\times[0,1]$. Our results are analogs of those in [1].

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Continual distribution with acceleration and condensation flows

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The kinetic equation Boltzmann is the main instrument to study the complicated phenomena in the multiple-particle systems, in particular, rarefied gas. This kinetic integro-differential equation for the model of hard spheres has a form [1, 2]:

$$D(f) = Q(f, f), \tag{1}$$

$$D(f) = \frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x}, \tag{2}$$

$$\begin{aligned} Q(f, f) = \frac{d^2}{2} \int\limits_{\mathbb{R}^3} dv_1 \int\limits_{\Sigma} d\alpha |(v - v_1, \alpha)| [f(t, v_1', x) f(t, v', x) - \\ - f(t, v_1, x) f(t, v, x)], \end{aligned} \tag{3}$$

We will consider the continual distribution [3]:

$$f = \int\limits_{\mathbb{R}^3} \varphi(t, x, u) M(v, u, x, t) du, \tag{4}$$

which contains the local Maxwellian of special form describing the acceleration and condensation flows of a gas (is an analogue of vortices) [4]. They have the form:

$$M(v, u, x, t) = \rho_0 e^{\beta\left((u-[\omega\times t])^2+2[\omega\times x]\right)} \left(\frac{\beta}{\pi}\right)^{\frac{3}{2}} e^{-\beta(v-u-[\omega\times t])^2}. \tag{5}$$

The purpose is to find such a form of the function $\varphi(t, x, u)$ and such a behavior of all hydrodynamical parameters so that the the uniform-integral (mixed) or pure integral remainder [3, 5], i.e. the functionals of the form:

$$\Delta = \sup_{(t,x)\in\mathbb{R}^4} \int\limits_{\mathbb{R}^3} |D(f) - Q(f, f)| dv, \tag{6}$$

$$\Delta_1 = \int\limits_{R^1} dt \int\limits_{R^3} dx \int\limits_{R^3} |D(f) - Q(f, f)| dv, \tag{7}$$

become vanishingly small.

Also some sufficient conditions to minimization of remainder Δ or Δ_1 are found. The obtained results are new and may be used with the study of evolution of screw and whirlwind streams.

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On a flower-shape geometry

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Several important problems arising in many research fields, such as physics and differential geometry, lead to consider elliptic equations when a lack of compactness occurs. From the mathematical point of view, the main interest relies on the fact that often the tools of nonlinear functional analysis, based on compactness arguments, cannot be used, at least in a straightforward way, and some new techniques have to be developed.

Aim of the talk is to present some of these techniques, which strongly use symmetry, together with their applications to elliptic problems with a variational structure. In particular we deal with a group theoretical scheme, raised in the study of problems which are invariant with respect to the action of orthogonal subgroups, and we present a construction, called flower-shape geometry, and its applications to the study of nonlinear problems set in strip-like domains. These results appeared in a joint paper with Giuseppe Devillanova (Politecnico di Bari) and Giovanni Molica Bisci (Urbino).

On equicontinuity of families of mappings with one normalization condition by the prime ends

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Borel function $\rho : \mathbb{R}^n \to [0, \infty]$ is called *admissible* for Γ, abbr. $\rho \in \operatorname{adm}\Gamma$, if $\int\limits_\gamma \rho(x)|dx| \geqslant 1$ for each (locally rectifiable) $\gamma \in \Gamma$. We define the quantity

$$M(\Gamma) = \inf_{\rho \in \operatorname{adm}\Gamma} \int\limits_{\mathbb{R}^n} \rho^n(x)\, dm(x) \tag{1}$$

and call $M(\Gamma)$ *a modulus* of Γ; here m stands for the n-dimensional Lebesque measure, see [1, 6.1].

Given sets E and F and a domain D in $\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$, we denote $\Gamma(E, F, D)$ the family of all paths $\gamma : [0, 1] \to \overline{\mathbb{R}^n}$ joining E and F in D, that is, $\gamma(0) \in E$, $\gamma(1) \in F$ and $\gamma(t) \in D$ for all $t \in [0, 1]$.

An *end* of a domain D is an equivalence class of chains of cross-cuts of D. We say that an end K is a *prime end* if K contains a chain of cross-cuts $\{\sigma_m\}$, such that

$$\lim_{m\to\infty} M(\Gamma(C,\sigma_m,D))=0$$

for some continuum C in D. Set $\mathbb{B}^n:=\{x\in\mathbb{R}^n : |x|<1\}$. We say that the boundary of a domain D in $\mathbb{R}^n$ is *locally quasiconformal* if every point $x_0\in\partial D$ has a neighborhood U that admit a conformal mapping φ onto the unit ball $\mathbb{B}^n\subset\mathbb{R}^n$ such that $\varphi(\partial D\cap U)$ is the intersection of $\mathbb{B}^n$ and a coordinate hyperplane, see e.g. [2], cf. [3]. We say that a bounded domain D in $\mathbb{R}^n$ is *regular* if D may be mapped quasiconformally onto a bounded domain with a locally quasiconformal boundary. If $\overline{D}_P$ is the completion of a regular domain D by its prime ends and g_0 is a quasiconformal mapping of a domain D_0 with locally quasiconformal boundary onto D, then this mapping naturally determines the metric $\rho_0(p_1,p_2)=\left|\widetilde{g_0}^{-1}(p_1)-\widetilde{g_0}^{-1}(p_2)\right|$, where $\widetilde{g_0}$ is the extension of g_0 onto $\overline{D_0}$. Let $x_0\in\overline{D}$, $x_0\ne\infty$, $S(x_0,r)=\{x\in\mathbb{R}^n : |x-x_0|=r\}$, $A=A(x_0,r_1,r_2)=\{x\in\mathbb{R}^n : r_1<|x-x_0|<r_2\}$.

Let $f:D\to\mathbb{R}^n$, $n\geqslant 2$, and let $Q:\mathbb{R}^n\to[0,\infty]$ be a Lebesgue measurable function such that $Q(y)\equiv 0$ for $y\in\mathbb{R}^n\setminus f(D)$. Let $A=A(y_0,r_1,r_2)$ and let $\Gamma_f(y_0,r_1,r_2)$ denotes the family of all paths $\gamma:[a,b]\to D$ such that

$$f(\gamma)\in\Gamma(S(y_0,r_1),S(y_0,r_2),A(y_0,r_1,r_2))\,,$$

i.e., $f(\gamma(a))\in S(y_0,r_1)$, $f(\gamma(b))\in S(y_0,r_2)$, and $f(\gamma(t))\in A(y_0,r_1,r_2)$ for any $a<t<b$.

We say that, f *satisfies the inverse Poletsky inequality* at a point $y_0\in f(D)$, if the relation

$$M(\Gamma_f(y_0,r_1,r_2))\leqslant\int\limits_{f(D)\cap A(y_0,r_1,r_2)} Q(y)\cdot\eta^n(|y-y_0|)\,dm(y) \tag{2}$$

holds for any Lebesgue measurable function $\eta:(r_1,r_2)\to[0,\infty]$ satisfying the relation

$$\int\limits_{r_1}^{r_2}\eta(r)\,dr\geqslant 1\,. \tag{3}$$

We say that the boundary of D is *weakly flat* at a point $x_0\in\partial D$ if, for every number $P>0$ and every neighborhood U of the point x_0, there is a neighborhood $V\subset U$ such that $M(\Gamma(E,F,D))\geqslant P$ for all continua E and F in D intersecting ∂U and ∂V. We say that the boundary ∂D is weakly flat if the corresponding property holds at every point of the boundary.

Given domains $D,D'\subset\mathbb{R}^n$, $n\geqslant 2$, points $a\in D$, $b\in D'$ and a Lebesgue measurable function $Q:D'\to[0,\infty]$ denote $\mathfrak{S}_{a,b,Q}(D,D')$ a family of all open discrete and closed mappings f of D onto D', satisfying the relation (2) for any $y_0\in D'$, while $f(a)=b$. The following statement holds.

Theorem 1. *Assume that, D has a weakly flat boundary, any component of which does not degenerate into a point. If $Q\in L^1(D')$ and D' is regular, then any $f\in\mathfrak{S}_{a,b,Q}(D,D')$ has a continuous extension $\overline{f}:\overline{D}\to\overline{D'}_P$, $\overline{f}(\overline{D})=\overline{D'}_P$, and, in addition, a family $\mathfrak{S}_{a,b,Q}(\overline{D},\overline{D'})$ which consists of all extended mappings $\overline{f}:\overline{D}\to\overline{D'}_P$, is equicontinuous in $\overline{D}$.*

The result mentioned above is published in [4].

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Equiaffine immersions of codimension two with flat connection

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We consider the affine immersions by K. Nomizu, T. Sasaki [1], namely $f : (M^n, \nabla) \to \mathbb{R}^{n+2}$. For a transversal frame $\xi_1,\ \xi_2$ and tangent vector fields $X,\ Y$ we have the affine analogues of Gauss and Weingarten decompositions, namely

$$D_X f_*(Y) = f_*(\nabla_X Y) + h^\alpha(X,\ Y)\xi_\alpha,$$

$$D_X \xi_\alpha = -f_*(S_\alpha X) + \tau_\alpha^\beta(X)\xi_\beta,$$

where h^α are components of the affine fundamental form, S_α are shape operators, τ_α^β are forms of transversal connection (with respect to $\xi_1,\ \xi_2$).

The *Weingarten mapping* $S_x : Q_x \times T_x(M^n) \to T_x(M^n)$ is defined [2] as follows: $(\xi, X) \mapsto S_\xi X$ at every point $x \in M^n$ (where $T_x(M^n)$ and Q_x are tangent and transversal distributions.)

For an affine immersion $f : (M^n, \nabla) \to \mathbb{R}^{n+2}$ with a transversal frame $\{\xi_1,\ \xi_2\}$, an *induced volume element* θ on M^n is defined [1, 3, 4] as follows:

$$\theta(X_1, \dots, X_n) = \det(f_*(X_1), \dots, f_*(X_n), \xi_1, \xi_2).$$

The transversal distribution Q with frame $\{\xi_1,\ \xi_2\}$ is called *equiaffine* if $\nabla_X \theta = 0$ for all $X \in T_x(M^n), x \in M^n$. For two-codimension affine immersion this condition is equivalent [4] to

$$\tau_1^1(X) + \tau_2^2(X) \equiv 0.$$

With an equiaffine transversal distribution Q we have an *equiaffine structure* $(\nabla,\ \theta)$ on M^n.

We will consider an affine immersion $f : (M^n, \nabla) \to \mathbb{R}^{n+2}$ with flat connection ∇ and equiaffine transversal distribution. Two-codimensional affine surfaces with different additional properties have been studied by many authors. Flat affine surfaces in $\mathbb{R}^4$ with flat normal connection were studied in [3]. The description of a parallel affine immersions $(M^n, \nabla) \to \mathbb{R}^{n+2}$ with flat connection in dependence on the rank of the Weingarten mapping were given in [2].

Let us remind that in general case (codimension k) the kernel and the image of the Weingarten mapping is defined by $\ker S = \bigcap\limits_{\alpha=1}^{k} \ker S_\alpha,\ \operatorname{im} S = \bigcup\limits_{\alpha=1}^{k} \operatorname{im} S_\alpha$. We say that Weingarten mapping is p-dimensional if $\operatorname{rank} S := \dim \operatorname{im} S = p$. It was proved [6] that for the immersion

$f:(M^n,\nabla)\to\mathbb{R}^{n+k}$ (for $k<n$) with maximal pointwise codimension and flat connection ∇ the following relations hold true:

$$1)\dim\ker S\geq n-k;\qquad 2)\ker h\subseteq\ker S;\qquad 3)\dim\operatorname{im}\,S\leq k;$$

$$4)\text{ if }\ \dim\operatorname{im}\,S=k,\ \text{ then }\ \dim\ker S=n-k\ \text{ and }\ \ker h=\ker S.$$

It was also proved [6] that the distribution $\mathcal{S}$ of the kernels of Weingarten mapping is integrable on M^n and there exists a transversal distribution which is stationary along the leaves of the foliation $\mathcal{FS}$.

Since in the case of codimension two we have $\dim\operatorname{im}\,S\leq 2$, $\dim\ker S\geq n-2$, so we have only three possible values for the dimension of im S, namely $0,\ 1,\ 2$. The most studied are affine immersions with zero and two-dimensional Weingarten mapping.

It is well known that an affine immersion with zero Weingarten mapping ($S\equiv 0$) has a flat connection and it is affinely equivalent to the graph of certain smooth map $F:M^n\to\mathbb{R}^2$ (see for example [5, 1, 6]), i. e.

$$f:(u^1,\ldots,\ u^n)\mapsto(u^1,\ldots,\ u^n,\ f^1(u^1,\ldots,\ u^n),\ f^2(u^1,\ldots,\ u^n)).$$

Obviously, a graph immersion is equiaffine.

According to the properties which were discussed in [6], in case $\dim\operatorname{im}\,S=2$ we obtain $\ker h=\ker S$ and $\dim\ker h=n-2$. Therefore such a submanifold is a submanifold of rank two (by the rank of Gaussian (Grassmann) mapping) [7]. Rank-two submanifold is a ruled submanifold with $(n-2)$-dimensional rulings over a two-dimensional base. In case this submanifold is a cylinder, its connection is determined by the connection of the cylinder base. In the general case the problem on its connection remains open.

We obtain a parametrization of a submanifold with one-dimensional Weingarten mapping and given properties. Such a submanifold is a peculiar "mix" of a graph and a ruled submanifold.

The main result. *Let $f:(M^n,\nabla)\to\mathbb{R}^{n+2}$ be an affine immersion with rank two affine fundamental form, equiaffine structure, flat connection ∇, one-dimensional Weingarten then there exists three types of its parametrization:*

(i) $\vec{r}=g(u^1,\ldots,u^n)\vec{a}_1+\int\vec{\varphi}(u^1)du^1+\sum\limits_{i=2}^{n}u^i\vec{a}_i;$

(ii) $\vec{r}=(g(u^2,\ldots,u^n)+u^1)\vec{a}+\int v(u^1)\vec{\eta}(u^1)du^1+\sum\limits_{i=2}^{n}u^i\int\lambda_i(u^1)\vec{\eta}(u^1)du^1;$

(iii) $\vec{r}=(g(u^2,\ldots,u^n)+u^1)\vec{\rho}(u^1)+\int(v(u^1)-u^1)\dfrac{d\vec{\rho}(u^1)}{du^1}du^1+\sum\limits_{i=2}^{n}u^i\int\lambda_i(u^1)\dfrac{d\vec{\rho}(u^1)}{du^1}du^1.$

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Some vanishing theorems of sufficient character about holomorphically projective mappings of Kahlerian spaces on the whole

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The generalized Bochner technique (see, for example, [1]) allows to broad to the noncompact but compete Kahlerian spaces some well-known theorems of holomorphically projective unique definability that have been proved previously only to the compact ones (see, for example, [2]). In particular, the next statements are true.

Theorem 1. *Complete connected Kahlerian C^r-spaces K^n $(n > 2,\ r > 3)$ with positive definite Ricci form don't admit non-trivial (different from affine) holomorphically projective mappings on the whole.*

Corollary 2. *Complete connected Kahlerian C^r-spaces K^n $(n > 2,\ r > 3)$ that have sign-definite metric form sign of which coincides with the sign of scalar curvature don't admit non-trivial (different from affine) holomorphically projective mappings on the whole.*

Corollary 3. *Complete connected Kahlerian C^r-spaces K^n $(n > 2,\ r > 3)$ that have positively definite metric form and non-positively definite on the set of symmetric tensors b^{ij} form*

$$R_{\alpha\gamma\sigma\beta}b^{\alpha\beta}b^{\gamma\sigma}$$

don't admit non-trivial (different from affine) holomorphically projective mappings on the whole.

Examples of Kahlerian spaces of considered types are known. In particular, complete connected Kahlerian C^r-spaces K^n $(n > 2,\ r > 3)$ of constant non-positive holomorphic curvature with positively definite metric form satisfies conditions of the both corollaries.

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Investigation of the connection between different models of topologies on a finite set

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One of the unsolved problems of discrete mathematics is the problem of counting all topologies on a finite set. Topologies on a finite set were modeled using various mathematical objects (graphs, partial order relations, Boolean functions and their normal forms, $(0,1)$-matrices of a special form, etc.). In [1] topologies were studied using the topology vector, the concept of which was introduced in [2]. In [3], in addition to the topology vector, Boolean functions and a maximal 2-CNF were used. The question arises about the relationship between the objects of different models, which can be used both to continue research and to verify the results. This issue was partially raised by us in our work [5].

In this paper, we consider the connection between models in the form of $(0,1)$-matrices of a special form and in the form of ordered sets $(M_1, M_2, ..., M_n)$ of minimal neighborhoods of elements of a given ordered finite set $X = (x_1, x_2, ..., x_n)$ (using such sets, one can instantly pass to topology vectors - ordered sets of integers α_k, which were effectively used in [1,2]).

According to [4] the $(0,1)$-matrix σ_{ij}, where $1 \leq i, j \leq n$, corresponds to some topology on X (in this case, this matrix is called a grid, and its order is the order of the grid) if and only if the following conditions are true:

1) $\sigma_{ij} = 1$, if $x_i \in \overline{x}_j$,

2) $\sigma_{ij} = 0$ in the other case

(Here the symbol $\overline{x}_j$ indicates the closure of a point x_j in a given topology).

Let, in the i-line of the matrix $\sigma_{ir_1} = 1$,$\sigma_{ir_2} = 1$,...,$\sigma_{ir_k} = 1$, the other elements be equal to zero. From $\sigma_{ir_1} = 1$ follows $x_i \in \overline{x}_{r_1}$, and then $M_i \cap \{x_{r_1}\} \neq \varnothing$. So, $x_{r_1} \in M_i$. Similarly, with $\sigma_{ir_2} = 1$ we get $x_{r_2} \in M_i$ etc. Hence, $M_i \supseteq \{x_i, x_{r_1}, ..., x_{r_k}\}$. On the other hand, for an element x_p from M_i the inclusion $x_i \in \overline{x}_p$ is obvious. Then $\sigma_{ip} = 1$. Thus, the i-line of the $(0,1)$-matrix corresponds to the minimal neighborhood of the element x_i.

The connection found made it possible to prove some properties of networks using minimal neighborhoods. In particular, the following properties of networks:

1) $\sigma_{ii} = 1$ at all $i = 1, ..., n$;

2) if $\sigma_{ir} = 1$ and $\sigma_{rj} = 1$, then $\sigma_{ij} = 1$;

3) The network σ defines T_0-topology on X (is T_0-network) if and only if $\sigma_{ij}\sigma_{ji} = 0$ at $i \neq j$.

Using these and other properties of networks and their connection with sets of minimal neighborhoods (bases of topologies), we enumerate all possible networks of T_0-topologies on a 4-element set and find the total number of T_0-topologies and the number 355 all of the topologies on this set using the well-known formula from work [6].

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Normal subgroups of iterated wreath products of symmetric groups and alternating with symmetric groups

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In this research we continue our previous investigation of wreath product normal structure [1].

The lattice of normal subgroups and their properties for finite iterated wreath products $S_{n_1} \wr \ldots \wr S_{n_m}$, $n, m \in \mathbb{N}$ are found. Special classes of normal subgroups and their orders and generators are found. Further, the monolith of these wreath products is investigated by us.

Let $k(\pi)$ be the number of cycles in decomposition of permutation π of degree n.

The number $n - k(\pi)$ is denoted by $dec(\pi)$, and is called a decrement [2] of permutation π.

As well known [2] the minimal number of transpositions in factorization of a permutation π on transpositions is happen to be equal to $dec(\pi)$. We set $dec(e) = 0$. Therefore the decrement of n-cycle is $n-1$.

If $\pi_1, \pi_2 \in S_n$, then the following formula holds:

$$dec(\pi_1 \cdot \pi_2) = dec(\pi_1) + dec(\pi_2) - 2m, m \in \mathbf{N}, \tag{1}$$

where m is number of joint simplifying transpositions in π_1 and π_2.

The trivial subgroup of S_n we denote by E.

Definition 1. The set of elements from $S_n \wr S_n, n \geqslant 5$ or $n = 3$ of the tableaux form: $[e]_1, [a_1, a_2, \ldots, a_n]_2$, satisfying the following condition

$$\sum_{i=1}^{n} dec([a_i]_2) = 2k, k \in \mathbb{N}, \tag{2}$$

we will call set of type $\widetilde{A}^{(2)}$ and denote this set by $E \wr \widetilde{A}_n$. For brevity of notation this subgroup be also denoted by $\widetilde{A}_n^{(2)}$. It follows directly from the definition that the set of these elements supplemented by the operation of multiplication in the subdirect product, coincides with the group $E \rtimes (\underbrace{S_n \boxtimes S_n \boxtimes S_n \ldots \boxtimes S_n}_{n})$, where subdirect product satisfies to condition (2).

We remind that the intersection of all non-trivial normal subgroups $Mon(G)$ of G is called the monolith of a group G.

Proposition 2. *Elements of first type form the subgroup $e \wr A_n$. This subgroup is the* ***monolith of*** *$S_n \wr S_n$.*

Now we can recursively construct easiest and elegant subgroup $E \wr \widetilde{A}_n^{(2)}$ of $S_n \wr S_n \wr S_n$.

Definition 3. The subgroup $E \wr \widetilde{A}_n^{(2)}$ be denoted by $\widetilde{A}_n^{(3)}$.

The order of $E \wr \widetilde{A}_n^{(2)}$ is $(n!)^{3n} : 2^3$. Furthermore we prove that $E \wr \widetilde{A}_n^{(2)} \lhd S_n \wr S_n \wr S_n$.
Let the set of elements from $S_n \wr S_n \wr S_n, n \geqslant 3$ of the form:

$$[e]_1, [e, e, \ldots, e]_2, [a_1, a_2, \ldots, a_{n^2}]_3$$

satisfying the following condition

$$\sum_{i=1}^{n^2} dec([a_i]_3) = 2k, k \in \mathbb{N}, \tag{3}$$

be denoted by $\widetilde{A}_{n^2}^{(3)}$.

Proposition 4. *The set of elements of type $\widetilde{A}_{n^2}^{(3)}$ forms a subgroup in $S_n \wr S_n \wr S_n$. Moreover $\widetilde{A}_{n^2}^{(3)} \lhd S_n \wr S_n \wr S_n$.*

Remark 5. We note that $\widetilde{A}_n^{(3)} < \widetilde{A}_{n^2}^{(3)}$. The order of $\widetilde{A}_{n^2}^{(3)}$ is $(n!)^{n^2} : 2$. Furthermore $\widetilde{A}_n^{(3)} \lhd S_n \wr S_n \wr S_n$.

Definition 6. *A subgroup in $S_n \wr S_n$ is called $\widetilde{\widetilde{T_n}}$ if it consists of:*

1) *elements of $E \wr A_n$,*
2) *elements with the tableau [3] presentation $[e]_1,\ [\pi_1, \ldots, \pi_n]_2$, that $\pi_i \in S_n \setminus A_n$.*

One easy can validates a correctness of this definition, i.e. that the set of such elements form a subgroup and its normality. This subgroup has structure $\tilde{T}_n \simeq (\underbrace{A_n \times A_n \times \cdots \times A_n}_{n}) \rtimes C_2 \simeq \underbrace{S_n \boxplus S_n \ldots \boxplus S_n}_{n}$, where the operation of a subdirect product $\boxplus$ is subject to items 1) and 2).

Definition 7. A subgroup in $S_n \wr S_n \wr S_n$ is of the type $\widetilde{T}_{n^2}^{(3)}$ if it consists of:

1) elements of the form $E \wr E \wr A_n$,
2) elements with the tableau [3] presentation $[e]_1,\ [e \ldots, e]_2,\ [\pi_1 \ldots, \pi_n, \pi_{n+1} \ldots, \pi_{n^2}]_3$, wherein $\forall i = 1, \ldots, n$: $\pi_i \in S_n \setminus A_n$.

We define recursively the subgroup $\widetilde{T}_n^{(3)}$ having n different intervals of elements with the same parity permutations on X^2.

Definition 8. The subgroup of $S_n \wr S_n \wr S_n$ having structure $E \wr \widetilde{T_n}$ is denoted by $\widetilde{T}_n^{(3)}$. The following isomorphism $\widetilde{T}_n^{(3)} \simeq \underbrace{S_n \boxplus S_n \ldots \boxplus S_n}_{n} \times \underbrace{S_n \boxplus S_n \ldots \boxplus S_n}_{n} \times \ldots \times \underbrace{S_n \boxplus S_n \ldots \boxplus S_n}_{n}$, where a tuple $S_n \boxplus S_n \ldots \boxplus S_n$ repeats n times, holds. The operation of a subdirect product $\boxplus$ is determined by Definition 6.

The operation $\boxplus$ accords with the properties described in item 1 and 2 of Definition 6, also $\boxplus$ is determined by automorphism in $\tilde{T}_n \simeq (\underbrace{A_n \times A_n \times \cdots \times A_n}_{n}) \rtimes C_2$ in this case.

Remark 9. Note that in $\widetilde{T}_n^{(3)}$ vertex permutation of tableau third part satisfy the condition: elements with the tableau presentation $[e]_1$, $[e \ldots, e]_2$, $[\pi_1 \ldots, \pi_n; \pi_{n+1} \ldots, \pi_{n^2}]_3$, that either all $\pi_i \in S_n \setminus A_n$ or all $[\pi_i]_3 \in A_n$ for $1 < i \leq n$, $n+1 \leq i < 2n$, ... , $n^2 - n < i \leq n^2$.

Here are the names of (almost all) predefined theorem-like environments.

Proposition 10. *The subgroup $E \wr A_n$ is the **monolith of** $S_n \wr S_n$.*

We call level of $AutX^*$ as active if it has at least one non-trivial permutation. Denote by $Aut_f X^*$ the group of all finite automorphism of spherically homogeneous rooted tree.

Proposition 11. *Let $H \lhd Aut_f X^*$ with depth k then H contains k-th level subgroup P having all even vertex permutations $p_{ki} \in A_n$ on X^k and trivial permutations in vertices of rest of levels. Furthermore P is normal in W provided k is last active level of $Aut_f X^*$.*

Theorem 12. *Proper normal subgroups in $S_n \wr S_m$ (action of group is left), where $n, m \geq 3$ with $n, m \neq 4$ are of the following types:*

1) *the subgroups of the first level stabilizer [1, 4] are*

$$E \wr \widetilde{A_m}, \widetilde{T_m}, E \wr S_m, E \wr A_n,$$

2) *the subgroups that act on both levels are $A_n \wr \widetilde{A_m}$, $S_n \wr \widetilde{A_m}$, $A_n \wr S_m$,*

wherein the subgroup $S_n \wr \widetilde{A_m} \simeq S_n \leftthreetimes (\underbrace{S_m \boxtimes S_m \boxtimes S_m \ldots \boxtimes S_m}_{n})$ endowed with the subdirect product [4] satisfying to condition (3), moreover $S_n \wr \widetilde{A_m}$ has two isomorphic copies, embedded into $S_n \wr S_m$ in different ways.

In total there are 8 proper normal subgroups in $S_n \wr S_m$.

Proposition 13. *All normal subgroups of $S_n \wr (S_m \times S_k)$ can be partitioned in 2 types:*

1) *$E \wr (N_i \times N_j)$, where $N_i \lhd \prod_{k=1}^{n} S_m^{(k)}$ and $N_j \lhd \prod_{l=1}^{n} S_l^{(l)}$.*

2) *$\widetilde{A}_l \wr (N_i \times N_j)$, where $\widetilde{A}_i \lhd S_n$, N_i and N_j are subgroups from item 1) possessing an extension by $\widetilde{A}_i$ in a correspondent groups $S_n \wr S_m$ and in $S_n \wr S_k$. The full list of them: $S_n \wr \left(S_m \times \widetilde{A}_k\right)$, $S_n \wr \left(\widetilde{A}_m \times \widetilde{A}_k\right)$, $S_n \wr \left(\widetilde{A}_m \times S_k\right)$, also $A_n \wr \left(S_m \times \widetilde{A}_k\right)$, $A_n \wr \left(\widetilde{A}_m \times \widetilde{A}_k\right)$, $A_n \wr \left(\widetilde{A}_m \times S_k\right)$.*

We denote the set of normal subgroup of $S_n \wr S_n$ by $N(S_n \wr S_n)$. Subgroup with number i from $N(S_n \wr S_n)$ is denoted by $N_i(S_n \wr S_n)$.

Theorem 14. *The full list of normal subgroups of $S_n \wr S_n \wr S_n$ consists of 50 normal subgroups. These subgroups are the following:*

1) ***Type*** T_{023} *contains:* $E \wr \tilde{A}_n \wr H$, $\widetilde{T_n} \wr H$, *where* $H \in \{\tilde{A}_n, \tilde{A}_{n^2}, S_n\}$. *There are 6 subgroups.*
2) ***The second type of subgroups is subclass in*** T_{023} *with new base of wreath product subgroup* $\tilde{A}_{n^2}$*:* $E \wr S_n \wr \tilde{A}_{n^2}$, $E \wr A_n \wr \tilde{A}_{n^2}$ $E \wr N_i(S_n \wr S_n)$. *Therefore this class has 12 new subgroups. Thus, the total number of normal subgroups in* ***Type*** T_{023} *is 18.*
3) ***Type*** T_{003}*:* $A^{(3)}_{00(n^2)} = E \wr E \wr \tilde{A}_{n^2}$, $\widetilde{T_{n^2}}$, $\widetilde{T_n}^{(3)}$. *Hence, here are 3 new subgroups.*
4) ***Type*** T_{123}*:* $N_i(S_n \wr S_n) \wr S_n$, $N_i(S_n \wr S_n) \wr \tilde{A}_n$ *and* $N_i(S_n \wr S_n) \wr \tilde{A}_{n^2}$. *Thus, there are 29 new normal subgroups in* T_{123}*, taking into account repetition* [5].

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Asymptotic behavior of the widths of classes of the generalized Poisson integrals

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Let L_p, $1 \le p \le \infty$, and C be the spaces of 2π–periodic functions with standard norms $\|\cdot\|_{L_p}$ and $\|\cdot\|_C$, respectively.

Denote by $C^{\psi}_{\bar\beta,p}$, $1 \le p \le \infty$, the set of all 2π-periodic functions f, representable as convolution

$$f(x) = \frac{a_0}{2} + \frac{1}{\pi}\int_{-\pi}^{\pi} \varphi(x-t)\Psi_{\bar\beta}(t)dt, \quad a_0 \in \mathbb{R}, \quad \varphi \in B^0_p = \{g \in L_p : \ \|g\|_p \le 1, \ g \perp 1\}, \tag{1}$$

with a fixed generated kernel $\Psi_{\bar\beta} \in L_{p'}$, $1/p + 1/p' = 1$, the Fourier series of which has the form

$$S[\Psi_{\bar\beta}](t) = \sum_{k=1}^{\infty} \psi(k)\cos\left(kt - \frac{\beta_k \pi}{2}\right), \quad \beta_k \in \mathbb{R}, \quad \psi(k) \ge 0.$$

A function f in the representation (1) is called $(\psi, \bar\beta)$-integral of the function φ and is denoted by $\mathcal{J}^{\psi}_{\bar\beta}\varphi$ $(f = \mathcal{J}^{\psi}_{\bar\beta}\varphi)$. If $\psi(k) \ne 0$, $k \in \mathbb{N}$, then the function φ in the representation (1) is called

$(\psi,\bar\beta)$-derivative of the function f and is denoted by $f^{\psi}_{\bar\beta}$ ($\varphi = f^{\psi}_{\bar\beta}$). The concepts of $(\psi,\bar\beta)$-integral and $(\psi,\bar\beta)$-derivative was introduced by Stepanets [1]. Since $\varphi\in L_p$ and $\Psi_{\bar\beta}\in L_{p'}$, then the function f of the form (1) is a continuous function, i.e. $C^{\psi}_{\bar\beta,p}\subset C$ (see [1, Proposition 3.9.2.]).

In the case $\beta_k\equiv\beta,\ \beta\in\mathbb{R}$, the classes $C^{\psi}_{\bar\beta,p}$ are denoted by $C^{\psi}_{\beta,p}$.

For $\psi(k)=k^{-r}, r>0$, the classes $C^{\psi}_{\bar\beta,p}$ and $C^{\psi}_{\beta,p}$ are denoted by $W^r_{\bar\beta,p}$ and $W^r_{\beta,p}$, respectively. The classes $W^r_{\beta,p}$ are the well-known Weyl-Nagy classes (see [1]). In other words, $W^r_{\beta,p}, 1\le p\le\infty$, are the classes of 2π-periodic functions f, representable as convolutions of the Weyl-Nagy kernels $B_{r,\beta}(t)=\sum\limits_{k=1}^{\infty}k^{-r}\cos\left(kt-\frac{\beta\pi}{2}\right),\ r>0,\ \beta\in\mathbb{R},$ with functions $\varphi\in B^0_p$.

If $r\in\mathbb{N}$ and $\beta=r$, then the functions $B_{r,\beta}$ are the well-known Bernoulli kernels and the corresponding classes $W^r_{\beta,p}$ coincide with the well-known classes W^r_p which consist of 2π-periodic functions f with absolutely continuous derivatives $f^{(k)}$ up to $(r-1)$-th order inclusive and such that $\|f^{(r)}\|_p\le 1$.

For $\psi(k)=e^{-\alpha k^r},\ \alpha>0,\ r>0$, the classes $C^{\psi}_{\bar\beta,p}$ and $C^{\psi}_{\beta,p}$ are denoted by $C^{\alpha,r}_{\bar\beta,p}$ and $C^{\alpha,r}_{\beta,p}$, respectively. The sets $C^{\alpha,r}_{\beta,p}$ are well-known classes of the generalized Poisson integrals [1], i.e. classes of convolutions with the generalized Poisson kernels

$$P_{\alpha,r,\beta}(t)=\sum_{k=1}^{\infty}e^{-\alpha k^r}\cos\left(kt-\frac{\beta\pi}{2}\right),\quad \alpha>0,\quad r>0,\quad \beta\in\mathbb{R}.$$

Further, let K be a convex centrally symmetric subset of C and let B be a unit ball of the space C. Let also F_N be an arbitrary N-dimensional subspace of space C, $N\in\mathbb{N}$, and $\mathscr{L}(C,F_N)$ be a set of linear operators from C to F_N. By $\mathscr{P}(C,F_N)$ denote the subset of projection operators of the set $\mathscr{L}(C,F_N)$, that is, the set of the operators A of linear projection onto the set F_N such that $Af=f$ when $f\in F_N$. The quantities

$$b_N(K,C)=\sup_{F_{N+1}}\sup\{\varepsilon>0:\varepsilon B\cap F_{N+1}\subset K\},$$

$$d_N(K,C)=\inf_{F_N}\sup_{f\in K}\inf_{u\in F_N}\|f-u\|_C,$$

$$\lambda_N(K,C)=\inf_{F_N}\inf_{A\in\mathscr{L}(C,F_N)}\sup_{f\in K}\|f-Af\|_C,$$

$$\pi_N(K,C)=\inf_{F_N}\inf_{A\in\mathscr{P}(C,F_N)}\sup_{f\in K}\|f-Af\|_C,$$

are called Bernstein, Kolmogorov, linear, and projection N-widths of the set K in the space C, respectively.

The results containing order estimates of the widths b_N, d_N, λ_N or π_N in the case of $K=C^{\psi}_{\bar\beta,p}$ (and, in particular, $W^r_{\beta,p}$ and $C^{\psi}_{\beta,p}$) can be found, for example, in the monographs of Tikhomirov, Pinkus, Kornejchuk, Romanyuk, Temlyakov etc.

Theorem 1. *Let $\bar\beta=\{\beta_k\}_{k=1}^{\infty}, \beta_k\in\mathbb{R}$, $\alpha>0$, $r>1$, $n\in\mathbb{N}$ and be such that*

$$(n-1)^r>\frac{1}{\alpha},\tag{2}$$

then the following inequalities hold

$$\frac{1}{\sqrt{\pi}}e^{-\alpha n^r}\left(1-\frac{2\gamma_{\alpha,r,n}e^{-2\alpha r(n-1)^{r-1}}}{1+2\gamma_{\alpha,r,n}e^{-2\alpha r(n-1)^{r-1}}}\right)^{\frac{1}{2}} \le P_{2n}(C^{\alpha,r}_{\bar\beta,2},C)$$

$$\le P_{2n-1}(C^{\alpha,r}_{\bar\beta,2},C) \le \frac{1}{\sqrt{\pi}}e^{-\alpha n^r}\left(1+e^{-2\alpha r n^{r-1}}\left(1+\frac{1}{2\alpha r n^{r-1}}\right)\right)^{\frac{1}{2}}, \qquad (3)$$

where P_N is any of the widths b_N, d_N, λ_N or π_N and

$$\gamma_{\alpha,r,n} = \left(1+\frac{1}{\alpha r(n-1)^{r-1}} + e^{-2\alpha(n-1)^r}\max\left\{e^{4\alpha}, \frac{e^2}{\alpha^{1+1/r}}\right\}\right). \qquad (4)$$

Theorem 2. *Let $\bar\beta = \{\beta_k\}_{k=1}^{\infty}, \beta_k \in \mathbb{R}$, $\alpha > 0$, $r > 1$, $n \in \mathbb{N}$ and the condition (2) is satisfied. Then as $n \to \infty$ the following asymptotic equalities hold*

$$\left.\begin{array}{l} P_{2n}(C^{\alpha,r}_{\bar\beta,2},C) \\ P_{2n-1}(C^{\alpha,r}_{\bar\beta,2},C) \end{array}\right\} = e^{-\alpha n^r}\left(\frac{1}{\sqrt{\pi}} + \mathcal{O}(1)\gamma_{\alpha,r,n}e^{-\alpha r(n-1)^{r-1}}\right), \qquad (5)$$

where P_N is any of the widths b_N, d_N, λ_N or π_N and $\gamma_{\alpha,r,n}$ is defined by (4) and $\mathcal{O}(1)$ are the quantities uniformly bounded in all parameters.

Note that the Theorem 2 complements the results of the works of Shevaldin (1992), Stepanets and Serdyuk (1995), Serdyuk (1999), Serdyuk and Sokolenko (2011), Serdyuk and Bodenchuk (2013), which contain exact estimates for the widths of the classes of convolutions with classical or generalized Poisson kernels.

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Poincaré-Reeb graphs of real algebraic domains

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An *algebraic domain* is a closed topological subsurface of a real affine plane whose boundary consists of disjoint smooth connected components of real algebraic plane curves. We study the geometric shape of an algebraic domain by collapsing all vertical segments contained in it:

this yields a *Poincaré–Reeb graph*, which is naturally transversal to the foliation by vertical lines. We show that any transversal graph whose vertices have only valencies 1 and 3 and are situated on distinct vertical lines can be realized as a Poincaré–Reeb graph.

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On univalent trinomials

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The Suffridge polynomials were introduced by T. Suffridge [1] and play an important role in complex analysis. Suffridge polynomials are closely related to the Brandt polynomials, first mentioned in M. Brandt's Ph.D. thesis [2] and rediscovered in [3].

The T-folded version of these polynomials were suggested in [4, 5] and several important conjectures about them were made.

In this talk we will outline the proof of these conjectures in the particular case of trinomials

$$z + az^{1+T} + bz^{1+2T}.$$

A beautiful geometry behind the scenes will be illuminated.

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On K-ultrametrics and $*$-measures

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The notion of K-ultrametric is introduced in [1]. A metric d on a set X is called a K-ultrametric, where $K \in [0, \infty]$, if $d(x, y) \leq K$, whenever $\min\{d(x, z), d(y, z)\} \leq K$.

Any 0-ultrametric is a metric, and any ∞-ultrametric is an ultrametric.

Some resent results are devoted to the K-ultrametrization of various functorial constructions on the category of K-ultrametric spaces: hyperspaces, spaces of probability measures, spaces of idempotent measurers [1, 2].

The aim of the talk is provide a construction of K-ultrametrization of the spaces of $*$-measures. Recall that a t-norm is a binary operation $*$ on $[0, 1]$ which is associative, commutative, continuous, monotone, and 1 is a unit for it.

A functional $\mu : C(X, [0, 1]) \to [0, 1]$ is called an $*$-measure if

1) μ preserves constants;
2) $\mu(\max\{\varphi, \psi\}) = \max\{\mu(\varphi), \mu(\psi)\}$;
3) $\mu(\lambda * \varphi) = \lambda * \mu(\varphi)$.

It is proved that the mentioned construction determines a functor on the category of K-ultrametric spaces and K-nonexpanding maps.

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The Iwasawa invariants of Z_p^d-covers of links

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In this talk, we will define the Iwasawa invariants of links and give two asymptotic formulae for the first homology groups of $\mathbb{Z}_p^d$-covers of links in rational homology 3-spheres, which are generalizations of the Iwasawa type formulae proven by Hillman–Matei–Morishita and Kadokami–Mizusawa. We will also provide examples of these formulae. Moreover, when $d = 2$, considering the twisted Whitehead links, we will explain that Iwasawa μ-invariants can be arbitrary non-negative integers. This is a joint work with Jun Ueki.

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The Riemann-Hilbert problem and holomorphic bundles framed along a real hypersurface

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The Riemann sphere $\mathbb{P}^1_{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ decomposes as the union $\mathbb{P}^1_{\mathbb{C}} = \bar{U}^- \cup \bar{U}^+$ of two closed disks $\bar{U}^- = \bar{D}$, $\bar{U}^+ = \mathbb{P}^1 \setminus D$ intersecting along their boundary $\partial \bar{U}^{\pm} = S^1$. The Riemann-Hilbert problem, as stated by Hilbert in [2, Kapitel X], asks:

The Riemann-Hilbert Problem. Let $\Gamma : S^1 \to \mathrm{GL}(r, \mathbb{C})$ be a smooth map. Find the pairs (Y^-, Y^+) of continuous maps $Y^{\pm} : \bar{U}^{\pm} \to \mathbb{C}^r$ which are holomorphic on $U^{\pm}$ and satisfy the condition $Y^+|_{S^1} = \Gamma Y^-|_{S^1}$.

More generally, consider

(1) A representation $\rho : G \to \mathrm{GL}(V)$ of a complex Lie group G on a finitely dimensional complex vector space V,
(2) A map $\Gamma : S \to G$ of class $\mathcal{C}^\kappa$ with $\kappa \in [0, \infty]$,
(3) An integer $m \in \mathbb{Z}$ and a V-valued polynomial $\gamma \in V[z]$. Put $d := \deg(\gamma) \in \mathbb{Z}_{\geq -1}$.

Regarding ∞ as an effective divisor on $\mathbb{P}^1_{\mathbb{C}}$, γ can be interpreted as an element of $H^0(\mathcal{O}(d\infty)_{(d+1)\infty} \otimes V)$. We ask:

The general RH problem on $\mathbb{P}^1$. Find the space of pairs (Y^-, Y^+) of continuous maps

$$Y^- : \bar{U}^- \to V,\ Y^+ : \bar{U}^+ \setminus \{\infty\} \to V$$

with Y^- holomorphic on U^-, Y^+ holomorphic on $U^+ \setminus \{\infty\}$ such that $Y^+_S = \rho(\Gamma) Y^-_S$, and

$$\lim_{z \to \infty} (z^{d-m} Y^+(z) - \gamma(z)) = 0$$

The geometric interpretation of the latter condition: Y^+ extends as a section of the sheaf $\mathcal{O}(m\infty) \otimes_{\mathbb{C}} V$ on U^+ whose image in $H^0(\mathcal{O}(m\infty)_{(d+1)\infty} \otimes V)$ via the obvious morphism is $z^{m-d} \otimes \gamma$. Hilbert's original problem is obtained taking ρ to be the canonical representation of $\mathrm{GL}(r, \mathbb{C})$ on $\mathbb{C}^n$, $m = 0$, and $\gamma = 0$.

Complex geometric point of view: Consider the sheaf $\mathcal{V}^\Gamma$ of *local solutions* of the RH problem with $m = \gamma = 0$; this sheaf is given explicitly by:

$$W \mapsto \left\{ \begin{pmatrix} f^- \\ f^+ \end{pmatrix} \in \begin{array}{c} \mathcal{C}^0(W \cap \bar{U}^-, V) \\ \times \\ \mathcal{C}^0(W \cap \bar{U}^+, V) \end{array} \middle| \begin{array}{c} f^+|_{W \cap S} = \rho(\Gamma)\, f^-|_{W \cap S}, \\ f^{\pm} \text{ is holomorphic on } W \cap U^{\pm} \end{array} \right\}.$$

Theorem 1. *Suppose $\kappa \in (1, \infty] \setminus \mathbb{N}$. The sheaf of $\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}$-modules $\mathcal{V}^\Gamma$ is locally free of rank* $\dim(V)$ *and coincides with the apparently smaller sheaf*

$$\left\{ \begin{pmatrix} f^- \\ f^+ \end{pmatrix} \in \begin{array}{c} \mathcal{C}^\kappa(W \cap \bar{U}^-, V) \\ \times \\ \mathcal{C}^\kappa(W \cap \bar{U}^+, V) \end{array} \middle| \begin{array}{c} f^+|_{W \cap S} = \rho(\Gamma)\, f^-|_{W \cap S}, \\ f^{\pm} \textit{ is holomorphic on } W \cap U^{\pm} \end{array} \right\}.$$

We have an obvious identification

$$H^0(\mathcal{O}(d\infty)_{(d+1)\infty} \otimes V) \xrightarrow{\simeq} H^0(\mathcal{V}^\Gamma(d\infty)_{(d+1)\infty}),$$

so γ gives an element $\nu_\gamma^\Gamma \in H^0(\mathcal{V}^\Gamma(d\infty)_{(d+1)\infty})$.

Consider the short exact sequence of coherent sheaves on $\mathbb{P}^1_{\mathbb{C}}$

$$0 \to \mathcal{V}^\Gamma((m-d-1)\infty) \to \mathcal{V}^\Gamma(m\infty) \to \mathcal{V}^\Gamma(m\infty)_{(d+1)\infty} \to 0$$

and the associated cohomology long exact sequence.

Corollary 2. (1) *The space of solutions of the general RH problem is non-empty if and only if the image of $z^{m-d} \otimes \nu_\gamma^\Gamma$ via the connecting morphism $H^0(\mathcal{V}^\Gamma(m\infty)_{(d+1)\infty}) \to H^1(\mathcal{V}^\Gamma((m-d-1)\infty))$ vanishes.*

(2) *If this is the case, this space has the natural structure of an affine space with model space $H^0\big(\mathcal{V}^\Gamma((m-d-1)\infty)\big)$, and can be identified with the pre-image of $z^{m-d} \otimes \nu_\gamma^\Gamma$ via the morphism $H^0\big(\mathcal{V}^\Gamma(m\infty)\big) \to H^0(\mathcal{V}^\Gamma(m\infty)_{(d+1)\infty})$.*

(3) *(Regularity) Any solution (Y^-, Y^+) of a RH problem with Γ of class $\mathcal{C}^\kappa$ is also of class $\mathcal{C}^\kappa$ up to the boundary.*

Note that, by Grothendieck's classification theorem, $\mathcal{V}^\Gamma$ splits as a direct sum of invertible sheaves, so $\mathcal{V}^\Gamma \simeq \bigoplus_{j=1}^r \mathcal{O}(n_j)$ with $n_j \in \mathbb{Z}$ and $\sum_{j=1}^r n_j = \deg(\mathcal{V}^\Gamma)$. For the canonical representation of $\mathrm{GL}(r,\mathbb{C})$ on $\mathbb{C}^r$ we have $\deg(\mathcal{V}^\Gamma) = -\deg(\det(\Gamma))$.

The RH problem can be naturally generalized in the framework of Riemann surfaces as follows: we replace $\mathbb{P}^1_{\mathbb{C}}$ by an arbitrary closed Riemann surface X, the circle S^1 by an arbitrary (not necessarily connected, not necessarily separating) oriented closed curve $S \subset X$, and Γ by a map $\Gamma \in \mathcal{C}^\kappa(S, G)$. Let $\widehat{X}_S$ be the Riemann surface with boundary obtained by cutting X along S. The unknown of the RH problem associated with these data is a meromorphic map $Y : \widehat{X}_S \setminus \partial\widehat{X}_S \dashrightarrow V$, which extends continuously around $\partial\widehat{X}_S$ and satisfies a compatibility condition associated with Γ. In this general framework one also has a complex geometric interpretation of the space of solutions which generalizes Corollary 2.

All these results are applications of a general gluing theorem for holomorphic bundles. The same theorem can be used to prove an isomorphism theorem between moduli spaces of framed holomorphic bundles [5], [1]:

Let E be a differentiable vector bundle of rank r on a closed complex manifold X, $S \subset X$ a closed, separating real hypersurface, $X = \bar{X}^- \cup \bar{X}^+$ the corresponding decomposition of X as union of compact complex manifolds with boundary, and $E^\pm := E|_{\bar{X}^\pm}$.

Theorem 3. *The moduli space $\mathcal{M}_S(E)$ of S-framed holomorphic structures on E can be identified with the fibre product of the moduli spaces $\mathcal{M}_{\partial\bar{X}^\pm}(E^\pm)$ of boundary framed (formally) holomorphic structures on $E^\pm$ over the space of Cauchy-Riemann operators on the trivial bundle of rank r on S.*

Note that $\mathcal{M}_{\partial\bar{X}^\pm}(E^\pm)$ can be further identified with moduli spaces of boundary framed Hermitian-Einstein connections on $E^\pm$ using a version of the classical Kobayashi-Hitchin correspondence [3] for complex manifolds with boundary [1], [6].

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About some Steiner trees

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I will talk about the famous Steiner problem.

We denote by $\mathcal{H}^1$ the linear Haurdorff measure (roughly speaking, length).

Problem 1 (Euclidean Steiner problem)**.** Let C be a compact subset of $\mathbb{R}^d$. To find a closed S such that $S \cup C$ is connected and $\mathcal{H}^1(S)$ is minimal.

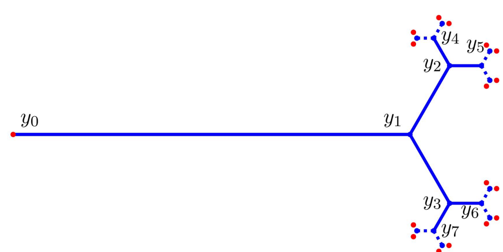
Some properties (if $\mathcal{H}^1(S) < \infty$) hold:

- S exists;
- S contains no loops;
- only two variants of neighbourhoods for points from $S \setminus C$;
- only two variants of a neighbourhood of a point $x \in S \setminus C$:
  - a regular tripod (x is a **branching point**);
  - a segment; x is an inner point.
- S contains at most countable number of branching points.

Usually S is usually called **Steiner tree**, and it is called **indecomposable** (irreducible, full), when $S \setminus C$ is connected. If C is totally disconnected then S should be connected.

Theorem 2 (Paolini–Stepanov–T, 2015; Cherkashin–T. 2023; Paolini–Stepanov 2023)**.** *There is a compact planar set C for which the unique solution of the Steiner problem is indecomposable and has infinite number of triple points.*

In the theorem C and Σ are self-similar fractals with a sufficiently small scale factor.



Theorem 3 (Basok, Cherkashin, T., 2022)**.** *In the plane for $m \geq 4$ the set of m terminals (considered as a subset of $\mathbb{R}^{2m}$) with non unique Steiner trees has the Hausdorff dimension $2m - 1$.*

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The multiplicities of non-acyclic SL_2-representations and L-functions of twisted Whitehead links

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We briefly survey a joint work [2] with Léo Bénard, Ryoto Tange, and Anh T. Tran, which is a continuation of our previous work [8] (See also [6, 9, 1, 7] and [4, 5, 3]).

We consider a natural divisor on $\mathrm{SL}_2\mathbb{C}$-character varieties of knots and links, given by the so-called acyclic Reidemeister torsion. We provide a geometric interpretation of this divisor. We focus on the particular family of odd twisted Whitehead links W_{2k-1}, where we show that this divisor has multiplicity two. Moreover, we apply these results to the study of the L-functions of the universal deformations of representations over finite fields of twisted Whitehead links.

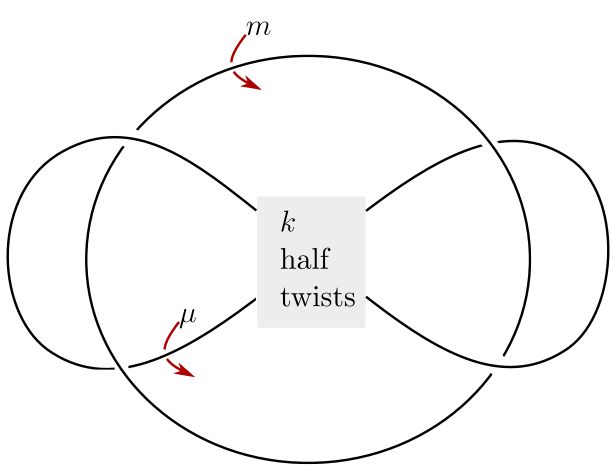

Figure 0.1. The twisted Whitehead link W_k

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Proximal connectedness. Spatially and descriptively connected spaces

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In this paper, we give results for spatially-connected spaces (X, δ) [5] (a widely-considered proximity space) and descriptively-connected spaces (X, δ_Φ), a recent form of proximity space [3] with a number of applications. Definition 1 is analogous to connectedness in digital topology [2].

Definition 1. Let (X, δ) be a proximity space. Then two nonempty subsets A and B are δ-connected, provided there exists a finite family of subsets $\{E_i\}_{i=0}^n$ of X such that $A = E_0$, $B = E_n$, and $E_i \ \delta \ E_{i+1}$ for all $i = 0, 1, \ldots, n-1$. A proximity space (X, δ) is said to be connected, provided any pair of subsets A, B of X are δ-connected.

Example 2. In Figure 1.1, let $h, k : I \to K$ be continuous maps called homotopies in a space K. A homotopic class for different maps h (denoted by $[h] = \left\{h_0, \ldots, h_i, \ldots, h_{n-1[n]}\right\}$ with $[n] = \bmod n \in \mathbb{Z}^+$) is a collection of $h_{i[n]}$ homotopic maps that have the same endpoints, namely, $h_i(0)$ and $h_i(1)$. The maps in $[h]$ are spatially near. Similarly, the maps in $[k]$ are spatially near. However, the homotopy classes $[h], [k]$ are spatially far. Also, from Definition 1,

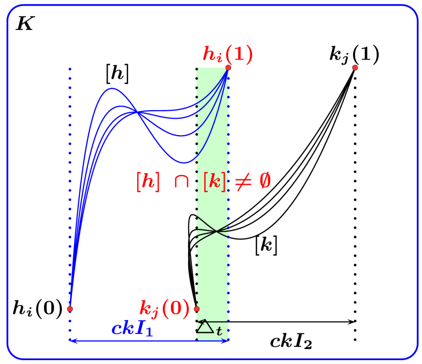

FIGURE 1.1. Spatially near homotopy maps in spatial far homotopy classes $[h], [k]$

every pair of nonempty subsets $A, B \in [h]$ are δ-connected, since $A\ \delta\ B$. Likewise, the maps in $[k]$ are δ-connected.

Theorem 3. *Let (X, δ_1) and (Y, δ_2) be proximity spaces. Then a map $f : X \to Y$ is proximally continuous if and only if a pair of δ_1-connected subsets of X is mapped to a pair of δ_2-connected subsets of Y.*

Definition 4. [1] Let (X, δ_Φ) be a descriptive proximity space. Then two subsets A and B are **δ_Φ-connected**, provided there exists a finite family of subsets $\{D_i\}_{i=0}^n$ of X such that $D_0 = A$, $D_n = B$, and $D_i\ \delta_\Phi\ D_{i+1}$ for all $i = 0, 1, \ldots, n-1$. A descriptive proximity space (X, δ_Φ) is said to be **descriptively connected**, provided any pair of subsets A, B of X are δ_Φ-connected.

Proposition 5. *In a digital topology space X, descriptively near sets in a digital image X are descriptively connected.*

Theorem 6. *Let (X, δ_{Φ_1}) and (Y, δ_{Φ_2}) be descriptive proximity spaces. Then a map $f : X \to Y$ is descriptive proximally continuous if and only a pair of δ_{Φ_1}-connected subsets of X is mapped to a pair of δ_{Φ_2}-connected subsets of Y.*

Given a set $X \subset \mathbb{Q} \times \mathbb{Q}$, a digital image on X is a map $Img : X \to \mathbb{R}$ so that each picture element $p \in X$ aka (sub)pixels or (sub)voxels has location $\mathbb{Q} \times \mathbb{Q}$ and value $Img(p) \in \mathbb{R}$. There are two distinct type digital images, namely, frames (denoted by Img_t or simply by fr_t) which is a time-ordered sequence of images in which each frame occurs at an elapsed time t in a video, and single images (denoted by Img). The intersection of the closures of bounded regions with nonempty interiors give rise to $\underset{\kappa}{\delta}$-connectedness.

Definition 7. Given a digital image Img, two subimages A and B, are adjacent, denoted by $A\ \underset{\kappa}{\delta}\ B$, provided there exist pixels $p \in A$ and $q \in B$ such that $p = q$ or p and q are adjacent.

Definition 8. A bounded region $E \subset fr$ with a non-empty interior is said to be $\underset{\kappa}{\delta}$-connected, provided for each pair of distinct voxels p and q in E, there exists a finite sequence of voxels $p = v_0, v_1, \ldots, v_n = q$ such that each pair of consecutive voxels v_i and v_{i+1} is $\underset{\kappa}{\delta}$-connected for all $i = 0, 1, \ldots, n-1$.

Proposition 9. *$\underset{\kappa}{\delta}$-connected regions are preserved under a continuous digital functions.*

Example 10. Consider a moving object appears as a bounded foreground region recorded in a video frame fr_t. Observe that the moving object -as a bounded region- is partitioned into (bounded) $\underset{\kappa}{\delta}$ connected subregions, which cover the moving object with a minimum number of contractible subregions is a form of geometric Lusternik-Schirel'mann category [4].

Spatially near subsets in a digital image (*i.e.*, subsets that share points) reside in a discrete proximity space.

Definition 11. For any pair of nonempty subsets A, B in a digital image Img, A is near B (denoted by $A\ \delta\ B$), provided A and B have points in common, *i.e.*, $A\ \delta\ B$ iff $A \cap B \neq \emptyset$. Hence, δ is a discrete proximity relation and (Img, δ) is a discrete proximity space [6, §40.2, pp. 266-267].

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p-Hyperbolic Zolotarev functions in boundary value problems for a p th order differential operator

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For the self-adjoint operator of the pth derivative, a system of fundamental solutions is constructed. This system is analogues to the classical system of sines and cosines. The properties of such functions are studied. Classes of self-adjoint boundary conditions are described. For the operator of the third derivative, the resolvent is calculated and an orthonormal basis of eigenfunctions is given.

Thinness at infinity and Deny's principle of positivity of mass in the theory of Riesz potentials

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This talk is based on [9], and it deals with the theory of potentials on $\mathbb{R}^n$, $n \geqslant 2$, with respect to the Riesz kernel $|x-y|^{\alpha-n}$, $\alpha \in (0,2]$, $\alpha < n$, where $|x-y|$ is the Euclidean distance between $x, y \in \mathbb{R}^n$. Denote by $\mathfrak{M}^+$ the cone of all positive Radon measures μ on $\mathbb{R}^n$ such that the Riesz *potential*

$$U^\mu(x) := \int |x-y|^{\alpha-n}\, d\mu(y)$$

is not identically infinite on $\mathbb{R}^n$, which according to [5, Section I.3.7] occurs if and only if

$$\int_{|y|>1} \frac{d\mu(y)}{|y|^{n-\alpha}} < \infty.$$

Then U^μ is actually finite everywhere on $\mathbb{R}^n$, up to a set of *zero Riesz capacity*, cf. [5, Section III.1.1].

The principle of positivity of mass was first introduced by J. Deny (see e.g. [2]), and for Riesz potentials it reads as follows [3, Theorem 3.11].

Theorem 1. *For any $\mu, \nu \in \mathfrak{M}^+$ such that*

$$U^\mu \leqslant U^\nu \quad \textit{everywhere on } \mathbb{R}^n, \tag{1}$$

we have $\mu(\mathbb{R}^n) \leqslant \nu(\mathbb{R}^n)$.

It is easy to verify that (1) can be slightly weakened by replacing 'everywhere on $\mathbb{R}^n$' by 'nearly everywhere on $\mathbb{R}^n$' (see [8, Theorem 2.6], establishing the principle of positivity of mass for potentials with respect to rather general function kernels on locally compact spaces). Recall that a proposition $\mathcal{P}(x)$ is said to hold *nearly everywhere* (*n.e.*) on $A \subset \mathbb{R}^n$ if $c_*(E) = 0$, where E is the set of all $x \in A$ for which $\mathcal{P}(x)$ fails, while $c_*(E)$ denotes the *inner Riesz capacity* of E, see [5, Section II.2.6].

The main result of this talk, given by Theorem 2, shows that Theorem 1 still holds even if (1) is fulfilled only on a proper subset A of $\mathbb{R}^n$, which however must be 'large enough' in an arbitrarily small neighborhood of $\infty_{\mathbb{R}^n}$, the Alexandroff point of $\mathbb{R}^n$. This discovery illustrates a special role of the point at infinity in Riesz potential theory, in particular with regard to the principle of positivity of mass.

Theorem 2. *Given $\mu, \nu \in \mathfrak{M}^+$, assume there exists $A \subset \mathbb{R}^n$ which is not inner α-thin at infinity, and such that*

$$U^\mu \leqslant U^\nu \quad \textit{n.e. on } A.$$

Then

$$\mu(\mathbb{R}^n) \leqslant \nu(\mathbb{R}^n).$$

Recall that according to [4, 7], $A \subset \mathbb{R}^n$ is said to be *inner α-thin at infinity* if

$$\sum_{k\in\mathbb{N}} \frac{c_*(A_k)}{q^{k(n-\alpha)}} < \infty, \tag{2}$$

where $q \in (1, \infty)$ and $A_k := A \cap \{x \in \mathbb{R}^n : \ q^k \leqslant |x| < q^{k+1}\}$; or equivalently, if either A is bounded, or $x = 0$ is an inner α-irregular boundary point for the inverse of A with respect to $|x| = 1$. (For the concept of inner α-irregular points for arbitrary $A \subset \mathbb{R}^n$ and relevant results, see [6, Section 6]; compare with [5, Section V.1], where A was required to be Borel.) We emphasize that *if A is not inner α-thin at infinity, then necessarily $c_*(A) = \infty$; but not the other way around* (see [7, Section 2]).

The following theorem shows that Theorem 2 is *sharp* in the sense that the requirement on A of not being α-thin at infinity can not in general be weakened.

Theorem 3. *If $A \subset \mathbb{R}^n$ is inner α-thin at infinity, then there are $\mu_0, \nu_0 \in \mathfrak{M}^+$ such that $U^{\mu_0} = U^{\nu_0}$ nearly everywhere on A, but nonetheless, $\mu_0(\mathbb{R}^n) > \nu_0(\mathbb{R}^n)$.*

Nevertheless, Theorem 2 remains valid for *arbitrary* $A \subset \mathbb{R}^n$ once we impose upon $\mu, \nu \in \mathfrak{M}^+$ suitable additional requirements (see Theorem 4 below).

A measure $\mu \in \mathfrak{M}^+$ is said to be *carried* by $A \subset \mathbb{R}^n$ if $\mathbb{R}^n \setminus A$ is μ-negligible, or equivalently if A is μ-measurable and $\mu = \mu|_A$, $\mu|_A$ being the trace of μ to A, cf. [1, Section V.5.7]. We denote by $\mathfrak{M}^+_A$ the cone of all $\mu \in \mathfrak{M}^+$ carried by A. (For closed A, μ is carried by A if and only if it is supported by A.)

A measure $\mu \in \mathfrak{M}^+$ is said to be *C-absolutely continuous* if $\mu(K) = 0$ for every compact set $K \subset \mathbb{R}^n$ of zero Riesz capacity. This certainly occurs if $\int U^\mu \, d\mu$ is finite (or, more generally, if U^μ is locally bounded); but not conversely, see [5, pp. 134–135].

Theorem 4. *For any set $A \subset \mathbb{R}^n$ and any C-absolutely continuous measures $\mu, \nu \in \mathfrak{M}^+_A$ such that $U^\mu \leqslant U^\nu$ n.e. on A, we still have $\mu(\mathbb{R}^n) \leqslant \nu(\mathbb{R}^n)$.*

Remark 5. If $A \cap A_I = \varnothing$, where A_I denotes the set of all inner α-irregular points for A, then the requirement of C-absolute continuity imposed on μ and ν, is unnecessary for the validity of Theorem 4.

Remark 6. The proofs of the above-quoted theorems are based on the theory of inner α-Riesz balayage as well as on that of inner α-Riesz equilibrium measures, both originated in [6, 7] (see also [8]). The concept of inner equilibrium measure is understood in an extended sense where its energy as well as its total mass may be infinite. The following two facts of these theories are crucial to our proofs:

- $A \subset \mathbb{R}^n$ is not α-thin at infinity if and only if the inner balayage of *any* $\mu \in \mathfrak{M}^+$ to A preserves its total mass (see [7, Corollary 5.3]).
- The inner α-Riesz equilibrium measure of $A \subset \mathbb{R}^n$ exists if and only if A is α-thin at infinity (see [7, Theorem 2.1]).

Remark 7. The results presented in the talk have already found applications to minimum Riesz energy problems in the presence of external fields, see for instance [10, Section 4.10].

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On the category of representations of a third-order semigroup without unit and zero elements

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The classification of the semigroups of third order (in terms of Cayley tables, up to isomorphism and antiisomorphism, was first received by T. Tamura in 1953 [1]. The minimal systems of generators and the corresponding defining relations for all such semigroups were described by V. M. Bondarenko and Y. V. Zatsikha in [2].

If one considers only commutative semigroups and only those that are not neither cyclic nor cyclic with an attached unit or zero element, then there exist, up to isomorphism, the following four semigroups (in square brackets are indicated all elements, in angular a minimal system of generators, and then the defining relations; the trivial relations for unit and zero generating elements 0 and e are not written out):

(a) $(0, b, c) = \langle b, c\rangle$: $b^2 = 0$, $c^2 = 0$, $bc = cb = 0$;

(b) $(0, b, c) = \langle b, c\rangle$: $b^2 = b$, $c^2 = c$, $bc = cb = 0$;

(c) $(0, b, c) = \langle b, c\rangle$: $b^2 = 0$, $c^2 = c$, $bc = cb = 0$;

(d) $(c^2, b, c) = \langle b, c\rangle$: $b^3 = b^2$, $c^3 = c$
(is a consequence of the remaining relations),
$b^2 = c^2$, $bc = cb = c$.

Except for the semigroup (a), all these semigroups are of finite representation type over any field K [2], and in this case one of the forms of studying the category of representations is the description of the Auslander algebra as the algebra of endomorphisms of the direct sum of representatives of all equivalence classes of indecomposable representations. In the simple case (b), the Auslander algebra was considered as an example in [3], and in the case (c) it was described in [4]. Here we consider the Auslander algebra of the semigroup (d) which is denoted by S_d.

Theorem 1. *The Auslander algebra $Aus_K(S_d)$ of the semigroup S_d over a field K of characteristic not equal to 2 is isomorphic to the algebra of all matrices of the form*

$$X = \begin{pmatrix} x_{11} & 0 & 0 & 0 & 0 \\ 0 & x_{22} & 0 & 0 & 0 \\ 0 & 0 & x_{33} & x_{34} & x_{35} \\ 0 & 0 & 0 & x_{33} & 0 \\ 0 & 0 & 0 & x_{54} & x_{55} \end{pmatrix},$$

where x_{ij} are elements of K.

Theorem 2. *The Auslander algebra $Aus_K(S_d)$ of the semigroup S_d over a field K of characteristic 2 is isomorphic to the algebra of all matrices of the form*

$$X = \begin{pmatrix} x_{11} & x_{12} & x_{13} & 0 & 0 & 0 \\ 0 & x_{11} & 0 & 0 & 0 & 0 \\ 0 & x_{32} & x_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & x_{44} & x_{45} & x_{46} \\ 0 & 0 & 0 & 0 & x_{44} & 0 \\ 0 & 0 & 0 & 0 & x_{65} & x_{66} \end{pmatrix},$$

where x_{ij} are elements of K.

These results were obtained together with Prof. V. M. Bondarenko. The first theorem was published in [5] and the second will be published in [6].

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Знаходження форми квантових графів за умов Діріхле на висячих вершинах

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Проблема існування коспектральних (або інакше ізоспектральних) графів виникла ще у минулому сторіччі. У класичній теорії графів коспектральними вважають неізоморфні графи з однаковим спектром матриці суміжності (див. [5], Розділ 6.1). У [4] був наведений перший приклад коспектральних графів.

У багатьох випадках більш важливу роль ніж матриця суміжності відіграє нормований лапласіан. Існують різні означення нормованого лапласіана, котрий ще називають дискретним лапласіаном (див. [7], С.2). Ми розуміємо під нормованим лапласіаном матрицю $D^{-1/2}AD^{-1/2}$, де A - матриця суміжності графа, а $D = diag\{d(v_1), d(v_2), ..., d(v_p)\}$ - матриця степенів вершин, де $d(v_j)$ - степінь вершини v_j.

У теорії квантових графів розглядають спектральні задачі, породжені рівняннями Штурма-Ліувілля на рівнобічних графах (метричних графах, з ребрами однакової довжини) з крайовими умовами Неймана або Діріхле на висячих вершинах і узагальненими умовами Неймана (умовами неперервності і Кірхгофа) у внутрішніх вершинах. Тут також виникає проблема коспектральності.

У [9] було показано, що існують коспектральні графи (неізометричні графи з однаковим спектром задачі Штурма-Ліувілля) у квантовій теорії графів. Слід зауважити, що у випадку графа з несумірними довжинами ребер спектр однозначно визначає форму графа [8].

Спектр задачі теорії квантових графів зв'язаний з нормованим лапласіаном відповідного
комбінаторного графа наступним чином: власні значення нормованого лапласіана взаємно
однозначно пов'язані з другими членами асимптотики власних значень задачі Штурма-Ліувілля з (узагальненими) умовами Неймана на вершинах цього графа (див. [3], де використані результати [2], [6] та [1]). Це дає змогу отримати інформацію про форму графа користуючись асимптотикою власних значень.

Для задачі Штурма-Ліувілля з умовами Неймана на висячих вершинах і умовами неперервності та Кіргофа у внутрішніх було доведено, що спектр однозначно визначає форму простого звязного рівнобічного графу, якщо кількість вершин не перевищує 5 і форму дерева, якщо кількість вершин не перевищує 8.

У даній роботі ми розглядаємо спектральну задачу Штурма-Ліувілля на простому зв'язному рівнобічному графі зі стандартними умовами у внутрішніх вершинах та умовами Діріхле на висячих вершинах (орієнтація ребер довільна):

$$-y_j'' + q_j(x)y_j = \lambda y_j, \quad (j = 1, 2, ..., g), \tag{1}$$

з умовами неперервності

$$y_j(0) = y_k(l) \tag{2}$$

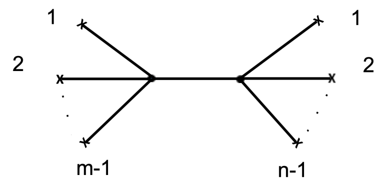

Рис 1. Граф подвійна зірка.

для всіх $j \in W^-(v_i)$, всіх $k \in W^+(v_i)$, де $W^+(v_i)$ множина індексів ребер, які входять у вершину v_i, $W^-(v_i)$ множина індексів ребер, що виходять із вершини v_i, умовами Кірхгофа

$$\sum_{k \in W^+(v_i)} y_k'(l) = \sum_{j \in W^-(v_i)} y_j'(0) \tag{3}$$

у внутрішніх вершинах та умовами Діріхле

$$y_j(0) = 0 \tag{4}$$

на висячих вершинах.

Отримано такий результат.

Теорема 1 *Нехай спектр* $\{\lambda_k\}_{k=1}^{\infty} = \bigcup_{i=1}^{s} \{\lambda_k^{(i)}\}_{k=1}^{\infty}$ *задачі (1)-(4), складається з підпослідовностей з асимптотикою*

$$\sqrt{\lambda_k^{(1)}} \underset{k\to\infty}{=} \frac{2\pi(k-1)}{l} + \gamma + O\left(\frac{1}{k}\right) \quad \text{для} \quad k \in \mathbb{N},$$

$$\sqrt{\lambda_k^{(2)}} \underset{k\to\infty}{=} \frac{2\pi k}{l} - \gamma + O\left(\frac{1}{k}\right) \quad \text{для} \quad k \in \mathbb{N},$$

$$\sqrt{\lambda_k^{(3)}} \underset{k\to\infty}{=} \frac{(2k-1)\pi}{l} + \gamma + O\left(\frac{1}{k}\right) \quad \text{для} \quad k \in \mathbb{N},$$

$$\sqrt{\lambda_k^{(4)}} \underset{k\to\infty}{=} \frac{(2k-1)\pi}{l} - \gamma + O\left(\frac{1}{k}\right) \quad \text{для} \quad k \in \mathbb{N},$$

$$\sqrt{\lambda_k^{(i)}} \underset{k\to\infty}{=} \frac{\pi k}{l} + O\left(\frac{1}{k}\right) \quad \text{для} \ \ i = 5, 6, ..., s \ \ \text{та} \ \ k \in \mathbb{N}.$$

Тоді ця асимптотика однозначно визначає форму графа як подвійної зірки з кількістями
периферичних ребер інцидентних з внутрішніми вершинами $m-1$ *та* $n-1$ *(див. рис. 1), де натуральні числа* m *і* n *становлять розв'язок системи рівнянь*

$$m + n = s - 1, \quad mn = (\cos\gamma l)^{-2}. \tag{5}$$

Ця система рівнянь має 2 розв'язки, котрі відповідають ізоморфним графам.

Доповідь грунтується на результатах статті [10], де, також, отримані теореми, подібні до теореми 1 для інших графів.

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Стійкість мінімальних поверхонь у субрімановому многовиді $\widetilde{E(2)}$

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Відомо, що у тривимірному евклідовому просторі повна зв'язна мінімальна поверхня є стійкою тоді й тільки тоді, коли є площиною. Цей результат був отриманий незалежно О.В. Погорєловим, М. до Кармо і К.К. Пенгом та Д. Фішер-Колбрі і Р. Шоеном (див., наприклад, [1]). Він узагальнює класичну теорему С.Н. Бернштейна, згідно з якою будь-яка повна явно задана мінімальна поверхня є площиною. У [2] було введене поняття мінімальної поверхні в субрімановому многовиді. У подальшому такі поверхні та їхня стійкість вивчалися для різних субріманових геометрій, див. огляд у [4]. Зокрема, у [3] були отримані результати типу Бернштейна, тобто опис стійких мінімальних поверхонь, у субрімановій тривимірній групі Гейзенберга.

Субрімановим многовидом зветься гладкий многовид M разом з гладким векторним розподілом $\mathcal{H}$ на M (він зветься горизонтальним розподілом) і гладким полем евклідових скалярних добутків $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ на $\mathcal{H}$ (субрімановою метрикою). Розглянемо гладку орієнтовану поверхню Σ у тривимірному субрімановому многовиді M, субріманова метрика $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ якого будується як обмеження на $\mathcal{H}$ деякої ріманової метрики M. Сингулярна множина Σ_0 цієї поверхні складається з тих точок $p \in \Sigma$, для яких дотична площина $T_p\Sigma$ збігається з $\mathcal{H}_p$. Якщо N – одиничне нормальне поле Σ у рімановому сенсі, то можно описати сингулярну множину як $\Sigma_0 = \{p \in \Sigma \mid N_h(p) = 0\}$, де N_h – ортогональна проекція поля N на

$\mathcal{H}$. Субріманова площа області $D \subset \Sigma$ визначається як $A(D) = \int_D |N_h| d\Sigma$, де $d\Sigma$ – ріманова форма площі Σ. Нормальною варіацією Σ, що задана гладкою функцією u, будемо називати відображення $\varphi\colon \Sigma \times I \to M$, що визначене умовою $\varphi_s(p) = \exp_p(su(p)N(p))$. Тут I – деякий окіл нуля в $\mathbb{R}$, а $\exp_p$ – ріманове експоненційне відображення. Іншими словами, ми будуємо варіацію традиційним для ріманової геометрії чином, випускаючи геодезичні з точки p в напрямку $u(p)N(p)$. Позначимо $A(s) = \int_{\Sigma_s} |N_h| d\Sigma_s$, де $\Sigma_s = \varphi_s(\Sigma)$. Тоді $A'(0)$ зветься першою варіацією площі, що відповідає φ, а $A''(0)$ – другою. Поверхня Σ називається мінімальною, якщо $A'(0) = 0$ для будь-яких нормальних варіацій з компактним носієм у $\Sigma \setminus \Sigma_0$. Зауважимо, що тут ми слідуємо рімановій традиції, називаючи мінімальними поверхнями стаціонарні точки субріманового функціонала площі. Мінімальна поверхня Σ називається стійкою, якщо $A''(0) \geq 0$ для будь-яких нормальних варіацій з компактним носієм у $\Sigma \setminus \Sigma_0$. У [3] було, зокрема, встановлено, що у субрімановій тривимірній групі Гейзенберга повна зв'язна мінімальна поверхня з порожньою сингулярною множиною є стійкою тоді й тільки тоді, коли ця поверхня є вертикальною евклідовою площиною.

Многовид $\widetilde{E(2)}$ визначається як універсальне накриття групи власних рухів площини. Це простір $\mathbb{R}^3$ з координатами (x, y, z) (де (x, y) відповідає паралельному перенесенню, а z – куту обертання), на якому структура групи Лі визначає наступний базис лівоінваріантних векторних полів:

$$X_1 = \cos z \frac{\partial}{\partial x} + \sin z \frac{\partial}{\partial y}, X_2 = \frac{\partial}{\partial z}, X_3 = \sin z \frac{\partial}{\partial x} - \cos z \frac{\partial}{\partial y}.$$

Розглянемо на $\widetilde{E(2)}$ ріманову метрику $g = \langle \cdot, \cdot \rangle$ таку, що $\{X_1, X_2, X_3\}$ є ортонормованим базисом в кожній точці. Зауважимо, що вона виявляється евклідовою. У якості горизонтального розподілу $\mathcal{H}$ візьмемо розподіл, що натягнутий на $\{X_1, X_2\}$, а у якості $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ – обмеження евклідової метрики на $\mathcal{H}$. Нехай Σ тепер – гладка орієнтована поверхня у $\widetilde{E(2)}$. Введемо деякі додаткові позначення. На регулярній частині $\Sigma \setminus \Sigma_0$ поверхні визначимо горизонтальне гаусове відображення $\nu_h = \frac{N_h}{|N_h|}$ та характеристичне векторне поле Z, яке у кожній точці утворюється з ν_h обертанням на прямий кут у площині $\mathcal{H}_p$ (в орієнтації, що визначена $N(p)$). Позначимо $S = \langle N, X_3 \rangle - |N_h| X_3 \in T_p\Sigma$. Векторне поле S доповнює Z до базису дотичної площини. Через ∇ позначатимемо ріманову коваріантну похідну. Нехай B – оператор Вейнгартена поверхні Σ відносно N, що визначається для будь-якого дотичного до Σ векторного поля W умовою $B(W) = -\nabla_W N$.

Теорема 1. *Нехай Σ – поверхня у $\widetilde{E(2)}$. Тоді перша нормальна варіація її площі, що задана функцією u, має наступний вигляд:*

$$A'(0) = \int_{\Sigma \setminus \Sigma_0} |N_h|^{-1} \left(- \langle B(Z), Z \rangle + \langle N, X_3 \rangle \langle \nabla_{\nu_h} X_3, \nu_h \rangle \right) u \, d\Sigma.$$

Якщо Σ мінімальна, то друга нормальна варіація її площі, що задана функцією u, має наступний вигляд:

$$A''(0) = \int_{\Sigma \setminus \Sigma_0} -2|N_h| \langle B(Z), S \rangle^2 u^2 + 2|N_h| \langle B(Z), Z \rangle \langle B(S), S \rangle u^2 +$$

$$+2|N_h| \langle B(Z), Z \rangle u^2 (\langle B(S), S \rangle + \langle B(Z), Z \rangle) - 2 \langle N, X_3 \rangle \langle B(S), Z \rangle Z(u) u +$$

$$+|N_h|^{-1} (Z(u) + \langle N, X_3 \rangle |N_h| \langle \nabla_{\nu_h} X_3, Z \rangle u)^2 -$$

$$-2|N_h|^2 \langle \nabla_{\nu_h} X_3, \nu_h \rangle u - |N_h|^3 \langle \nabla_{\nu_h} X_3, \nu_h \rangle^2 u^2 \quad d\Sigma.$$

Твердження 2. *Евклідова площина у $\widetilde{E(2)}$ є мінімальною тоді й тільки тоді, коли це горизонтальна або вертикальна площина. Усі мінімальні евклідові площини у $\widetilde{E(2)}$ є стійкими.*

Іншими прикладами мінімальних поверхонь є явно задані $y = A\cos z + B$ та $y = x + A(\sin z + \cos z) + B$, де A, B постійні. Наведені приклади демонструють, що з мінімальності поверхні у рімановому сенсі не випливає її субріманова мінімальність та навпаки. Будемо називати поверхню Σ у тривимірному субрімановому многовиді вертикальною, якщо $T_p\Sigma \perp \mathcal{H}_p$ для кожної $p \in \Sigma$. Зокрема, такі поверхні не містять сингулярних точок.

Теорема 3. *Будь-яка повна зв'язна вертикальна мінімальна поверхня у $\widetilde{E(2)}$ – це горизонтальна евклідова площина $z = C$ або паралельно перенесений уздовж площини (x, y) стандартний гелікоїд $x\cos z + y\sin z = 0$. При цьому гелікоїди є нестійкими.*

Звідси отримуємо наступний частковий результат типу Бернштейна.

Наслідок 4. *У $\widetilde{E(2)}$ повна зв'язна вертикальна мінімальна поверхня є стійкою тоді й тільки тоді, коли ця поверхня є горизонтальною евклідовою площиною.*

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Двовимірні неізотропні поверхні з плоскою нормальною зв'язністю і невиродженим грассмановим образом постійної кривини у просторі Мінковського

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Підмноговидами з плоскою нормальною зв'язністю у просторі Мінковського будемо, як і у будь-яких просторах постійної кривини, називати підмноговиди з нульовим тензором скруту. Такі підмноговиди з'являються як важливі приклади у багатьох дослідженнях, а також і самі виступають в якості об'єкта дослідження. Наприклад, у роботі [1] розглядаються занурення, образами яких є поверхні з плоскою нормальною зв'язністю, у роботах [2] і [3] досліджено двовимірні поверхні з плоскою нормальною зв'язністю, на яких геодезичні мають постійну кривину і нормальний скрут є нульовим відповідно. Коло задач диференціальної геометрії значно розширюється, коли починають використовувати грассмановий образ поверхні, який є узагальненням гауссового сферичного образу поверхні і важливою геометричною характеристикою поверхонь особливо в багатовимірних евклідових та неевклідових просторах. Об'єктом цього дослідження є неізотропні двовимірні поверхні з плоскою нормальною зв'язністю і невиродженим грассмановим образом у просторі Мінковського. Ми знайшли відповіді на такі питання: які значення може приймати кривина грассманового образу двовимірної поверхні з плоскою нормальною зв'язністю у чотиривимірному просторі Мінковського в залежності від типу поверхні та типу її грассманового образу; для яких значень k кривини грассманового образу існують поверхні з плоскою нормальною зв'язністю і грассмановим образом постійної кривини k. В роботі описуються всі неізотропні двовимірні поверхні з плоскою нормальною зв'язністю у просторі Мінковського, невироджений грассмановий образ яких має постійну кривину. Доведена

Теорема 1. *Для того, щоб двовимірна неізотропна поверхня V^2 класу C^k з плоскою нормальною зв'язністю у просторі Мінковського мала невироджений грассмановий образ постійної кривини k , необхідно й достатньо, щоб вона належала одному із наступних видів поверхонь:*

1) *Поверхні з постійною нульовою внутрішньою кривиною K і точковою коразмірністю 2;*
2) *Гіперповерхні тривимірних підпросторів з ненульовою внутрішньою кривиною K.*

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Геодезичні відображення симетричних просторів

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Псевдоріманів простір з метричним тензором називають *локально симетричним*, коли для кожної точки існує її окіл, в якому симетрія відносно цієї точки є автоморфізмом символів Христофеля.

Тензорною ознакою локально симетричних просторів (далі просто симетричних) є рівність нулю коваріантної похідної тензора Рімана. Використовуючи тензорну ознаку, М.С. Синюков довів, що симетричні псевдоріманові простори, відмінні від просторів сталої кривини, не допускають нетривіальних геодезичних відображень [1]. Цей результат багато разів дублювався та узагальнювався. Основними напрямками узагальнення були послаблення умов на тензор Рімана та накладання вимоги абсолютної паралельності на інші тензори.

Нами запропоновано два способи спеціалізації псевдоріманових просторів за аналогією з симетричними просторами:

- простори, в яких спеціальні тензори абсолютно паралельні за зв'язністю простору, на якій відображається даний [2];
- простори, в яких співпадають значення коваріантних похідних, обчислених за власною зв'язністю [3].

В першому випадку простори називаються *симетричними відносно відображення.* Якщо це відображення геодезичне, то такі простори називаються *геодезично симетричними просторами.*

Простори, які відповідають умовам другого способу спеціалізації, називаються *симетричними парами.*

Вивчені нетривіальні геодезичні відображення таких просторів. В обох випадках отримано вид лінійної форми основних рівнянь теорії геодезичних відображень.

Зокрема було доведено:

Теорема 1. [2] *Не існує геодезично Річчі симетричних просторів відмінних від просторів Ейнштейна.*

Теорема 2. [2] *Не існує геодезично симетричних псевдоріманових просторів відмінних від просторів сталої кривини.*

Теорема 3. [3] *Кожна симетрична пара псевдоріманових просторів є Вейль симетричною парою псевдоріманових просторів.*

Теорема 4. [3] *Вейль симетрична пара псевдоріманових просторів є гармонійною парою псевдоріманових просторів.*

Теорема 5. [3] *Геодезично симетричні пари можуть утворювати лише простори сталої кривини.*

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Про 3F-планарні відображення псевдо-ріманових з інтегровною структурою Яно-Хоу-Чена

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Досліджуючи майже контактні многовиди, К.Яно, С.Хоу і В.Чен [1] дійшли до поняття *квадриструктури*, структурний афінор якої задовольняє рівнянню $\phi^4 \pm \phi^2 = 0$.

Ми вивчаємо 3F-планарні відображення [2] псевдо-ріманових просторів (V_n, g_{ij}, F_i^h) і $(\overline{V}_n, \overline{g}_{ij}, \overline{F}_i^h)$ з інтегровною ермітовою афінорною структурою вказаного типу, основні ріняння яких в загальній за відображенням системі координат (x^i) мають вигляд:

$$\overline{\Gamma}_{ij}^h(x) = \Gamma_{ij}^h(x) + \sum_{s=0}^{3} \overset{s}{q}_{(i}(x) \overset{s}{F}{}_{j)}^h(x),$$

де

$$\overset{\circ}{F}{}_i^h = \delta_i^h, \quad \overset{1}{F}{}_i^h = F_i^h, \quad \overset{2}{F}{}_i^h = \overset{1}{F}{}_i^\alpha \overset{1}{F}{}_\alpha^h, \quad \overset{3}{F}{}_i^h = \overset{2}{F}{}_i^\alpha \overset{1}{F}{}_\alpha^h,$$

$$\overset{s}{F}{}_i^h(x) = \overset{s}{\overline{F}}{}_i^h(x),$$

$$F_\alpha^h F_\beta^\alpha F_\delta^\beta F_i^\delta + F_\alpha^h F_i^\alpha = 0,$$

$$g_{i\alpha} F_j^\alpha = -g_{j\alpha} F_i^\alpha,$$

$$F_{i,j}^h = F_{i|j}^h = 0,$$

Γ_{ij}^h, $\overline{\Gamma}_{ij}^h$ - компоненти об'єктів зв'язності V_n і $\overline{V}_n$, відповідно; $\overset{s}{q}_i(x)$ - деякі ковектори; F_i^h - афінор; $<,>$, $<|>$ - знаки коваріантної похідної в V_n і $\overline{V}_n$.

Ми довели, що за таких умов на афінор між векторами $\overset{s}{q}_i(x)$ є певний зв'язок. Аналіз цього зв'язку дає змогу довести, що є чотири типи 3F-планарних відображень з ермітовою інтегровною афінорною структурою Яно-Хоу-Чена: основний тип і три канонічних. Ми докладно вивчили один з канонічних типов , зокрема розглянули відображення цього типу (V_n, g_{ij}, F_i^h) на плоский простір і знайшли метрики всіх просторів, які допускають такі відображення, в спеціальній системі координат.

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Тополого-метрична теорія G-зображення чисел

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Нехай g_0–фіксоване число з проміжку $(0;\frac{1}{2}]$, $g_1 \equiv g_0 - 1$; $A \equiv \{0;1\}$ – алфавіт; $L \equiv A \times A \times ....$

Теорема 1. *Для будь–якого числа $x \in [0;g_0]$ існує послідовність $(\alpha_n) \in L$ така, що*

$$x = \alpha_1 g_{1-\alpha_1} + \sum_{k=2}^{\infty}\left(\alpha_k g_{1-\alpha_k}\prod_{i=1}^{k-1} g_{\alpha_i}\right) \equiv \Delta^G_{\alpha_1\alpha_2...\alpha_n...}.$$

Подання числа x рядом (95) називається його G–представленням, а символічний запис $\Delta^G_{\alpha_1\alpha_2...\alpha_n...}$ – G–зображенням. Майже всі числа відрізка $[0;g_0]$ (за винятком зліченної множини) мають єдине G–зображення і називаються G–унарними, а числа зліченної всюди щільної множини мають два зображення (вони називаються G–бінарними). Має місце рівність: $\Delta^G_{c_1...c_m01(0)} = \Delta^G_{c_1...c_m11(0)}$.

Специфічною особливістю G–зображення чисел є те, що оператор лівостороннього зсуву цифр G–зображення, означений рівністю $\omega(\Delta^G_{\alpha_1\alpha_2...\alpha_n...}) = \Delta^G_{\alpha_2\alpha_3...\alpha_n...}$, є неперервною коректно означеною функцією, а інверсор цифр $I(\Delta^G_{\alpha_1\alpha_2...\alpha_n...}) = \Delta^G_{[1-\alpha_1][1-\alpha_2]...[1-\alpha_n]...}$ є ніде не монотонною функцією необмеженої варіації.

Теорема 2. *Якщо $g_0 = \frac{1}{2}$, то має місце формула взаємозв'язку G–зображення і класичного двійкового зображення $\Delta^G_{\alpha_1\alpha_2...\alpha_n...} = \Delta^2_{0a_1a_2...a_n...}$,*

$$a_1 = \begin{cases} 0, & \text{коли } \alpha_1 = 0; \\ 1, & \text{коли } \alpha_1 = 1; \end{cases} a_{n+1} = \begin{cases} \alpha_{n+1}, & \text{коли } \alpha_1 + \ldots + \alpha_n - \text{ парне}, \\ 1-\alpha_{n+1}, & \text{коли } \alpha_1 + \ldots + \alpha_n - \text{ непарне}. \end{cases}$$

Теорема 3. *Якщо $g_0 = \frac{1}{2}$, то для будь–якого натурального числа a існує набір нулів та одиниць $(a_1, a_2, ..., a_n)$ такий, що $a = 2^n + \sum\limits_{k=1}^{n}[(-1)^{1+\sigma_k} a_k 2^{n-k}] \equiv (1a_1 \ldots a_n)_G$, де $\sigma_1 = 0$, $\sigma_k = a_1 + ... + a_{k-1}$, причому таких наборів існує рівно два.*

Теорема 4. *а) Якщо у G–зображенні натурального числа a більше цифр, ніж у G–зображенні натурального числа b, то $a \geq b$.*
б) Числа $a = (1a_1...a_{k-1}1a_{k+1}...a_n)_G$ і $b = (1a_1...a_{k-1}0b_{k+1}...b_n)_G$ перебувають у відношенні 1) $a \geq b$, якщо σ_k–непарне, 2) $a \leq b$, якщо σ_k–парне.

Доповідь присвячена геометрії G–зображення чисел (геометричному змісту цифр, властивостям циліндричних та хвостових множин) і результатам дослідження тополого–метричних і фрактальних властивостей множин $E_n(a) = \{x : \omega^n(x) \leq a = const\}$, $E_n = \{x : \omega^n(x) < x\}$, $E[G, \nu_0, \nu_1] = \{x = \Delta^G_{\alpha_1 \ldots \alpha_n \ldots}, \nu_1(x) = \lim\limits_{k\to\infty} k^{-1}(\alpha_1 + \ldots + \alpha_k), \nu_0(x) = 1 - \nu_1(x)\}$, $E[G, \nu_i(x)] = \{x : \nu_i(x)$– не існує$\}$.

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Наближення для просторів афінної зв'язності та індуковані відображення

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Розглянемо простір афінної зв'язності без скруту A_n, віднесений до довільної системи координат $\{x^1, x^2, \ldots, x^n\}$, з об'єктом зв'язності $\Gamma^h_{ij}(x)$; $M_0(x^h_0)$ — фіксована точка цього простору.

Побудуємо новий простір $\tilde{A}_n$, віднесений до координат $\{y^1, y^2, \ldots, y^n\}$, зі своїм об'єктом зв'язності $\tilde{\Gamma}^h_{ij}(y)$, який задається співвідношенням

$$\tilde{\Gamma}^h_{ij}(y) = -\frac{1}{3} \underset{0}{R}{}^h_{.(ij)l}\, y^l, \text{ де } \underset{0}{R}{}^h_{.ijl} = R^h_{.ijl}(M_0). \tag{1}$$

Вивчаються деякі геометричні об'єкти простору $\tilde{A}_n$. Зокрема, знайдено тензор Рімана:

$$\tilde{R}^h_{.ijk} = \underset{0}{R}{}^h_{.ijk} + \frac{1}{9}\left(R^\alpha_{.(ik)l_1} R^h_{.(\alpha j)l_2} - R^\alpha_{.(ij)l_1} R^h_{.(\alpha k)l_2}\right)\Bigg|_0 y^{l_1} y^{l_2}\,. \tag{2}$$

Згорнувши останнє співвідношення за індексами h та k, отримаємо тензор Річчі:

$$\tilde{R}_{ij} = \underset{0}{R}{}_{ij} + \frac{1}{9}\left(R_{il_1} R_{jl_2} + R^\alpha_{.(ij)l_1} R_{\alpha l_2}\right)\Bigg|_0 y^{l_1} y^{l_2}\,. \tag{3}$$

Підраховано компоненти параметрів Томаса:

$$\tilde{T}^h_{.ij} = -\frac{1}{3}\left[R^h_{.(ij)l} + \frac{1}{n+1}\left(R_{il}\delta^h_j + R_{jl}\delta^h_i\right)\right]\Bigg|_0 y^l\,. \tag{4}$$

Компоненти тензора проєктивної кривини (тензора Вейля):

$$\begin{gathered}\tilde{W}^h_{.ijk} = \underset{0}{W}{}^h_{.ijk} + \frac{1}{9}\Bigg[R^\alpha_{.(ik)l_1} R^h_{.(\alpha j)l_2} - R^\alpha_{.(ij)l_1} R^h_{.(\alpha k)l_2} \;-\; \frac{1}{n-1}\times \\ \times\left[\left(R_{il_1} R_{jl_2} + R^\alpha_{.(ij)l_1} R_{\alpha l_2}\right)\delta^h_k - \left(R_{il_1} R_{kl_2} + R^\alpha_{.(ik)l_1} R_{\alpha l_2}\right)\delta^h_j\right]\Bigg] y^{l_1} y^{l_2}.\end{gathered} \tag{5}$$

Далі розглядаються два простори афінної зв'язності: $\bar{A}_n$ з об'єктом зв'язності $\bar{\Gamma}^h_{ij}$ ($\bar{M}_0 \in \bar{A}_n$) і A_n з об'єктом зв'язності Γ^h_{ij}. Будуються їх наближення першого порядку — простори $\tilde{A}_n$ і $\tilde{\bar{A}}_n$. Вихідні простори допускають нетривіальне геодезичне відображення $\tilde{\gamma} : \tilde{\bar{A}}_n \to \tilde{A}_n$ у загальній системі координат $\{x^1, x^2, \ldots, x^n\}$.
Отримано тензор деформації $\tilde{P}^h_{ij} = \tilde{\bar{\Gamma}}^h_{ij} - \tilde{\Gamma}^h_{ij}$ відображення між просторами наближення:

$$\tilde{P}^h_{ij} = \varphi_{(i}\delta^h_{j)} + \frac{2}{3}\underset{0}{\psi}_{ij} y^h, \text{ де } \varphi_i = -\frac{1}{3}\underset{0}{\psi}_{il} y^l. \tag{7}$$

З'ясовано питання відносно властивості індукованого відображення $\tilde{\gamma} : \tilde{\bar{A}}_n \to \tilde{A}_n$.

Теорема 1. *Відображення просторів наближення $\tilde{\bar{A}}_n$ і $\tilde{A}_n$, яке індукується геодезичним відображенням вихідних просторів афінної зв'язності, не є геодезичним.*

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Закономірності квазі-геодезичних відображень узагальнено-рекурентно-параболічних просторів

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Розглянемо рекурентно-параболічний простір [1], [3] (V_n, g_{ij}, F^h_i), з метричним тензором $g_{ij}(x)$ та афінором $F^h_i(x)$, який допускає квазі-геодезичні відображення (КГВ) [2] на простір $(\overline{V}_n, \overline{g}_{ij}, \overline{F}^h_i)$. Тоді в загальній за відображенням системі координат (x^i) виконуються основні рівняння даного відображення [1]:

$$\overline{\Gamma}^h_{ij}(x) = \Gamma^h_{ij}(x) + \psi_{(i}(x)\delta^h_{j)} + \phi_{(i}(x)F^h_{j)}(x),$$
$$F^h_i = \overline{F}^h_i(x),$$
$$F^h_\alpha F^\alpha_i = 0,$$
$$g_{i\alpha}F^\alpha_j = -g_{j\alpha}F^\alpha_i, \quad \overline{g}_{i\alpha}F^\alpha_j = -\overline{g}_{j\alpha}F^\alpha_i,$$
$$F^h_{i,j} = F^h_{i|j} = q_j F^h_i,$$
$$i, h, j, ... = 1, 2, ..., n.$$

Тут $\Gamma^h_{ij}, \overline{\Gamma}^h_{ij}$ – компоненти об'єктів зв'язності $V_n, \overline{V}_n$, відповідно; ψ_i, ϕ_i, q_i – деякі коветори; «,» та «|» – знак коваріантної похідної в просторах $V_n, \overline{V}_n$, відповідно; дужками позначена операція симетрування.

Доведена [4]

Теорема 1. *Для того, щоб рекурентно-параболічний простір* (V_n, g_{ij}, F_i^h) *допускав нетрівіальне КГВ, необхідно і достатньо, щоб в ньому існував неособливий симетричний двічі коваріантний тензор* a_{ij}*, який задовольняє рівнянням*

$$a_{ij,k} = \lambda_\alpha F_i^\alpha g_{jk} + \lambda_\alpha F_j^\alpha g_{ik} + \lambda_i F_{jk} + \lambda_j F_{ik}, \tag{1}$$

і

$$a_{i\alpha} F_j^\alpha = -a_{j\alpha} F_i^\alpha, \quad det||a_{ij}|| \neq 0 \tag{2}$$

при деякому ковекторі $\lambda_i \neq 0$.

Питання про існування КГВ простору (V_n, g_{ij}, F_i^h) зводиться до дослідження диференціальних рівнянь (1) відносно вектора λ_i і тензора a_{ij}, який задовольняє (2).

Має місце

Теорема 2. *Для того, щоб псевдоріманові простір з інтегровною рекурентно-параболічною структурою* (V_n, g_{ij}, F_i^h) *допускав КГВ, необхідно і достатньо, щоб замкнена система диференціальних рівнянь у частинних похідних першого порядку типу Коші відносно функцій* a_{ij}, λ_i, ξ*:*

$$a_{ij,k} = \lambda_\alpha F_i^\alpha g_{jk} + \lambda_\alpha F_j^\alpha g_{ik} + \lambda_i F_{jk} + \lambda_j F_{ik},$$

$$\lambda_{i,l} = \frac{2}{n} a_{\alpha\beta} \tilde{\tilde{R}}_{il}^{\alpha\beta} - \lambda_i q_l + \frac{2}{n} \xi F_{il},$$

$$\xi_{,k} = a_{\alpha\beta} P_k^{\alpha\beta} + \lambda_\alpha \widetilde{T}_k^\alpha - 2\xi q_k,$$

мала нетривіальний розв'язок $a_{ij}(x)$, $\lambda_i(x) \neq 0$, $\xi(x)$, *який задовольняє умовам (2),* $\lambda_\alpha F_i^\alpha$ *– градієнтний вектор,* $a_{ij}(x) = a_{ji}(x)$. *Тут* $\tilde{\tilde{R}}_{il}^{hj}$, P_k^{hj}, $\widetilde{T}_k^h$ *виражаються через внутрішні об'єкти простору* V_n.

Дана теорема дає можливість звести дослідження існування КГВ до системи, яка може бути розв'язана за допомогою регулярних методів теорії диференціальних рівнянь.

Теорема 3. *Для того, щоб псевдоріманові простір з інтегровною рекурентно-параболічною структурою* (V_n, g_{ij}, F_i^h) *допускав КГВ, необхідно і достатньо, щоб система однорідних алгебраїчних рівнянь*

$$a_{\alpha\beta} S_{iklj}^{\alpha\beta} = 0,$$

$$a_{\alpha\beta} \tilde{P}_{ilk}^{\alpha\beta} + \lambda_\alpha \tilde{\tilde{T}}_{ilk}^\alpha = 0,$$

$$a_{\alpha\beta} Q_{ik}^{\alpha\beta} + \lambda_\alpha Q_{ik}^\alpha + \xi Q_{ik} = 0$$

та їх диференціальних продовжень в (V_n, g_{ij}, F_i^h) *мала нетривіальний розв'язок* $a_{ij}(x)$, $\lambda_i(x) \neq 0$, $\xi(x)$, *який задовольняє умовам (2),* $\lambda_\alpha F_i^\alpha$ *– градієнтний вектор,* $a_{ij}(x) = a_{ji}(x)$. *Тут* S_{iklj}^{hd} $\tilde{P}_{ilk}^{hj}$, $\tilde{\tilde{T}}_{ilk}^h$, Q_{ik}^{hj}, Q_{ik}^h, Q_{ik} *виражаються через внутрішні об'єкти простору* V_n.

Теореми 2 та 3 допомагають для будь-якого рекурентно-параболічного простору

$$(V_n, g_{ij}, F_i^h)$$

або знайти всі псевдоріманові простори, на які V_n допускає КГВ, або довести, що їх немає. Теореми 2 і 3 називають фундаментальними теоремами теорії КГВ рекурентно-параболічних просторів.

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Геометрія чисел у задачах конструктивної теорії локально складних функцій

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Неперервні функції з локально складною структурою тополого-метричного, інтегрально та диференціального, варіаційного та фрактального змісту не можуть бути аналітично заданими виразами зі скінченною кількістю бінарних операцій. Існують різні підходи до їх визначення, зокрема, метод ітераційних функцій, задання функції системою функціональних рівнянь, з використанням різних систем зображення чисел, з застосуванням перетворювачів цифр, проектування одного зображення в інше тощо.

Доповідь присвячена локально складним функціям, визначеним нескінченними системами функціональних рівнянь, залежним від нескінченної кількості дійсних параметрів. В класі розглядуваних функцій ніде не монотонні та ніде не диференційовні функції, функції канторівського типу, функції розподілу випадкових величин, абсолютно неперервні функції та сингулярні функції.

Розглядається чотири послідовності дійсних чисел: (Θ_n), (b_n), (p_n), (σ_n), які визначають нескінченну систему функціональних рівнянь

$$f(b_n + \Theta_n x) = \sigma_n + p_n f(x), n \in Z. \tag{1}$$

Остання система є основним об'єктом дослідження, результатам якого присвячена дана доповідь.

Теорема 1. *Якщо виконуються такі умови:*

1) $\Theta_{-n} = \Theta_n > 0$ *і* $\sum\limits_{n\in Z} \Theta_i = 1$*;*

2) $b_n = b_{n-1} + \Theta_{n-1} = \sum\limits_{i=-\infty}^{n-1} \Theta_i$*;*

3) $|p_i| < 1, \sum\limits_{i=\in Z} p_i = 1$*;*

4) $\sigma_n = \sigma_{n-1} + p_{n-1} = \sum\limits_{i=-\infty}^{n-1} p_i > 0,$

то система (1) *має у класі неперервних функцій, визначених на відрізку* $[0;1]$*, єдиний розв'язок.*

Зауваження 2. Далі вважається, що умови 1) — 4) для функції f, що задовольняє систему (1), виконуються. Якщо $p_i = \Theta_i$ для будь-якого $i \in Z$, то $f(x) = x$.

Теорема 3. *Якщо серед членів послідовності* (p_n) *немає від'ємних елементів, то* f *— функція розподілу на відрізку* $[0;1]$*.*

Теорема 4. *Якщо існує* $p_i = 0$*, то міра Лебега множини несталості (тобто доповнення до об'єднання інтервалів сталості) рівна нулю, а отже,* f *є функцією канторівського типу.*

Теорема 5. *Якщо* $f(x)$ *— функція канторівського типу, а* X *— випадкова величина, рівномірно розподілена на* $[0;1]$*, то випадкова величина* $Y = f(X)$ *має чисто дискретний розподіл.*

Теорема 6. *Якщо серед членів послідовності* (p_n) *існують від'ємні числа, то* f *є функцією необмеженої варіації.*

Теорема 7. *Якщо серед членів послідовності* (p_n) *немає нулів, але є від'ємні числа, то функція* f *є ніде не монотонною.*

Теорема 8. *Якщо серед членів послідовності* (p_n) *існують від'ємні числа і нулі, то* f *є функцією необмеженої варіації, яка не має проміжків монотонності за виключенням проміжків сталості.*

Лема 9. *Графік* Γ_f *функції* f *є структурно фрактальною множиною, а саме* N*-самоафінною множиною з наступною структурою самоафінності:*

$$\Gamma_f = \bigcup_{i=-\infty}^{+\infty} \Gamma_i, \quad \Gamma_i = f_i(\Gamma_f), \quad f_i = \begin{cases} x' = \Theta_i x + b_i, \\ y' = p_i y + \sigma_i. \end{cases}$$

Теорема 10. *Має місце рівність*

$$\int_0^1 f(x)dx = \frac{\sum\limits_{i\in Z} \sigma_i \Theta_i}{1 - \sum\limits_{i\in Z} \Theta_i p_i}.$$

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Розв'язок задачі Колмогорова-Нікольського для інтерполяційних поліномів Лагранжа на класах узагальнених інтегралів Пуассона

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Через $C^{\alpha,r}_{\beta,p}$, $\alpha > 0$, $r > 0$, $\beta \in \mathbb{R}$, $1 \leq p \leq \infty$, позначимо множину 2π–періодичних функцій $f(x)$, які при всіх $x \in \mathbb{R}$ можна представити у вигляді згортки

$$f(x) = \frac{a_0}{2} + \frac{1}{\pi}\int\limits_{-\pi}^{\pi} P_{\alpha,r,\beta}(x-t)\varphi(t)dt, \ a_0 \in \mathbb{R}, \quad \varphi \perp 1, \quad \varphi \in L_p, \quad ||\varphi||_p \leq 1, \tag{1}$$

з ядрами вигляду

$$P_{\alpha,r,\beta}(t) = \sum_{k=1}^{\infty} e^{-\alpha k^r} \cos\left(kt - \frac{\beta\pi}{2}\right), \quad \alpha, r > 0, \quad \beta \in \mathbb{R}.$$

Функцію f у рівності (1) називають узагальненим інтегралом Пуассона функції φ і позначають через $\mathcal{J}^{\alpha,r}_{\beta}\varphi$, з іншого боку функцію φ у рівності (1) називають узагальненою похідною функції f і позначають через $f^{\alpha,r}_{\beta}$ (тобто, $\varphi(\cdot) = f^{\alpha,r}_{\beta}(\cdot)$) [1].

Для будь–якої функції $f(x)$ із простору неперрвних 2π–періодичних функцій C через $\widetilde{S}_{n-1}(f;x)$ будемо позначати тригонометричний поліном порядку $n-1$, що інтерполює $f(x)$ у вузлах $x^{(n-1)}_k = \frac{2k\pi}{2n-1}$, $k \in \mathbb{Z}$, тобто такий, що

$$\tilde{S}_{n-1}(f; x^{(n-1)}_k) = f(x^{(n-1)}_k), \ k = 0, 1, ..., 2n-2. \tag{2}$$

Поліноми $\widetilde{S}_{n-1}(f;\cdot)$ однозначно задаються інтерполяційними умовами (2) і називаються інтерполяційними поліномами Лагранжа.

Позначимо через $\tilde{\rho}_n(f;\cdot)$ відхилення від функції $f \in C$ її інтерполяційного полінома Лагранжа $\tilde{S}_{n-1}(f;\cdot)$

$$\tilde{\rho}_n(f;x) = f(x) - \tilde{S}_{n-1}(f;x).$$

Мета нашого дослідження полягає в тому, щоб при всіх $x \in \mathbb{R}$, $\alpha > 0$, $\beta \in \mathbb{R}$, $r \in (0,1)$ і $1 \leq p \leq \infty$, знайти розв'язок задачі Колмогорова-Нікольського для інтерполяційних поліномів Лагранжа $\tilde{S}_{n-1}(f;x)$ вигляду (2) на класах узагальнених інтегралів Пуассона $C^{\alpha,r}_{\beta,p}$, тобто встановити асимптотичні при $n \to \infty$ рівності для величин

$$\tilde{\mathcal{E}}_n(C^{\alpha,r}_{\beta,p}; x) = \sup_{f \in C^{\alpha,r}_{\beta,p}} |\tilde{\rho}_n(f;x)| \, . \tag{3}$$

Має місце наступна теорема.

Теорема 1. *Нехай* $r \in (0,1)$, $\alpha > 0$, $\beta \in \mathbb{R}$, $1 \le p \le \infty$ *і* $x \in \mathbb{R}$. *Тоді при* $p = 1$ *і* $n \ge n_*(\alpha, r, 1)$

$$\tilde{\mathcal{E}}_n(C^{\alpha,r}_{\beta,1}; x) = e^{-\alpha n^r} n^{1-r} \left| \sin \frac{2n-1}{2} x \right| \left(\frac{2}{\pi \alpha r} + \delta^*_{n,1} \left(\frac{1}{n^{1-r}} + \frac{1}{(\alpha r)^2 n^r} \right) \right); \tag{4}$$

при $1 < p < \infty$ *і* $n \ge n_*(\alpha, r, p)$

$$\tilde{\mathcal{E}}_n(C^{\alpha,r}_{\beta,p}; x) = e^{-\alpha n^r} n^{\frac{1-r}{p}} \left| \sin \frac{2n-1}{2} x \right|$$
$$\times \left(\frac{2\|\cos t\|_{p'}}{\pi^{1+\frac{1}{p'}} (\alpha r)^{\frac{1}{p}}} F^{\frac{1}{p'}} \left(\frac{1}{2}, \frac{3-p'}{2}; \frac{3}{2}; 1 \right) + \delta^*_{n,p} \left(\left(1 + \frac{(\alpha r)^{\frac{p'-1}{p}}}{p'-1} \right) \frac{1}{n^{\frac{1-r}{p}}} + \frac{p^{\frac{1}{p'}}}{(\alpha r)^{1+\frac{1}{p}} n^r} \right) \right), \tag{5}$$

а при $p = \infty$ *і* $n \ge n_*(\alpha, r, \infty)$

$$\tilde{\mathcal{E}}_n(C^{\alpha,r}_{\beta,\infty}; x) = e^{-\alpha n^r} \left| \sin \frac{2n-1}{2} x \right| \left(\frac{8}{\pi^2} \ln \frac{n^{1-r}}{\alpha r} + \delta^*_{n,\infty} \right). \tag{6}$$

У формулах (4)–(6) *для величин* $\delta^*_{n,p} = \delta^*_{n,p}(\alpha, r, \beta, x)$ *виконується оцінка* $|\delta^*_{n,p}| < 40\pi^4$.

Оцінки (4)–(6) доповнюють результати робіт [2]–[4], де було знайдено розв'язок вказаної задачі Колмогорова–Нікольського на класах $C^{\alpha,r}_{\beta,p}$ при всіх $r \ge 1$, $\alpha > 0$, $\beta \in \mathbb{R}$ і $1 \le p \le \infty$.

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Про нижню оцінку діаметра образу круга

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Нехай задано сім'ю Γ кривих γ в комплексній площині $\mathbb{C}$. Борелеву функцію $\rho:\mathbb{C}\to[0,\infty]$ називають *допустимою* для Γ, пишуть $\rho\in adm\,\Gamma$, якщо $\int\limits_{\gamma}\rho(z)|dz|\geqslant 1$ для кожної кривої $\gamma\in\Gamma$. Нехай $p\in(1,\infty)$. Тоді *p-модулем* сім'ї Γ називається величина

$$\mathrm{M}_p(\Gamma)=\inf_{\rho\in adm\,\Gamma}\int\limits_{\mathbb{C}}\rho^p(z)\,dxdy\,.$$

Для довільних множин E, F, і G в $\mathbb{C}$, через $\Delta(E,F;G)$ позначимо сім'ю всіх кривих $\gamma:[a,b]\to\mathbb{C}$, які з'єднують E і F в G, тобто $\gamma(a)\in E$, $\gamma(b)\in F$ і $\gamma(t)\in G$ при $a<t<b$. Покладемо

$$\mathbb{A}(z_0,r_1,r_2)=\{z\in\mathbb{C}:r_1<|z-z_0|<r_2\},$$
$$S_i=S(z_0,r_i)=\{z\in\mathbb{C}:|z-z_0|=r_i\},\quad i=1,2.$$

Нехай D — область в комплексній площині $\mathbb{C}$ та $Q:D\to[0,\infty]$ — вимірна за Лебегом функція. Будемо говорити, що гомеоморфізм $f:D\to\mathbb{C}$ є *кільцевим Q-гомеоморфізмом відносно p-модуля в точці $z_0\in D$*, якщо співвідношення

$$\mathrm{M}_p(\Delta(fS_1,fS_2;fD))\leqslant\int\limits_{\mathbb{A}}Q(z)\,\eta^p(|z-z_0|)\,dxdy$$

виконується для будь-якого кільця $\mathbb{A}=\mathbb{A}(z_0,r_1,r_2)$, $0<r_1<r_2<d_0$, $d_0=\mathrm{dist}(z_0,\partial D)$, і для кожної вимірної функції $\eta:(r_1,r_2)\to[0,\infty]$ такої, що $\int\limits_{r_1}^{r_2}\eta(r)dr=1$.

Всюди далі будемо вважати, що $q_{z_0}(r)=\frac{1}{2\pi r}\int\limits_{S(z_0,r)}Q(z)|dz|$ — середнє інтегральне значення функції Q по колу $S(z_0,r)=\{z\in\mathbb{C}:|z-z_0|=r\}$.

Нижче наведено теорему про нижню оцінку діаметра образу круга.

Теорема 1. *Припустимо, що $Q\colon\mathbb{C}\to[0,\infty]$ — вимірна за Лебегом функція така, що середнє інтегральне значення $q_{z_0}(r)$ скінченне для м.в. $r>0$. Нехай $f:\mathbb{C}\to\mathbb{C}$ — кільцевий Q-гомеоморфізм відносно p-модуля в точці z_0 при $p>2$, де z_0 — деяка точка* *в* $\mathbb{C}$, $r_0 > 0$. *Тоді для будь-якого* $R > r_0$ *виконується оцінка*

$$\operatorname{diam}\left(fB(z_0,R)\right) \geqslant 2\left(\frac{p-2}{p-1}\right)^{\frac{p-1}{p-2}} \left(\int\limits_{r_0}^{R} \frac{dt}{t^{\frac{1}{p-1}}\, q_{z_0}^{\frac{1}{p-1}}(t)}\right)^{\frac{p-1}{p-2}},$$

де $B(z_0,R)=\{z\in\mathbb{C}\colon |z-z_0|\leqslant R\}$.

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