Geometric properties of interception curves

Yagub N. Aliyev

ADA University, Baku, Azerbaijan

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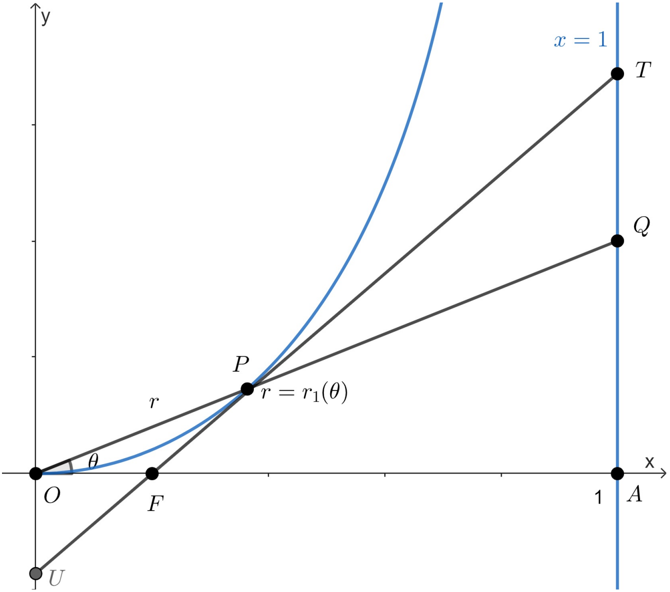
Planar Curve

Question 1. Suppose that two points $P(x, y)$ and Q, initially at $O(0, 0)$ and $A(1, 0)$, respectively, move with constant and equal velocities so that Q is on the line $x = 1$, and P is on the ray OQ. What curve is defined by the point P?
This problem appear in problems related to the interception of high-speed targets by beam rider missiles

Elnan, O.R.S.; Lo, H. Interception of High-Speed Target by Beam Rider Missile. *Aiaa J.* **1963**, *1*, 1637–1639. https://doi.org/10.2514/3.1870.
Vinh, N.X. Comment on "Interception of High-Speed Target by Beam Rider Missile". *AIAA J.* **1964**, *2*, 409. https://doi.org/10.2514/3.2316.

Plane curve



Polar coordinates

Since the speeds of the points P and Q are equal, the length of the curve OP and the length of the line segment AQ, which is $\tan\theta$, are equal for each θ. By using the well-known formula for the length of a curve $r = r(\theta)$, given in polar coordinates, we find that

$$\int_0^{\theta} \sqrt{r(t)^2 + (r'(t))^2}\,dt = \tan\theta. \tag{1}$$

By taking the derivative of both sides of (1) and simplifying, we obtain ODE

$$r(\theta)^2 + (r'(\theta))^2 = \frac{1}{\cos^4\theta}, \tag{2}$$

with initial condition $r(0) = 0$.

Cartesian coordinates

First, note that in the cartesian coordinates, (1) can be written as

$$\int_0^x \sqrt{1 + (y'(t))^2}\,dt = \frac{y}{x}. \tag{3}$$

By taking the derivative of both sides of (3) and simplifying, we obtain

$$x^2\sqrt{1 + (y'(x))^2} = y'x - y, \tag{4}$$

which in particular agrees with $y(0) = 0$.

Kamke, E. *Differentialgleichungen Lösungsmethoden und Lösungen*; Springer, Wiesbaden, Germany, 1977. https://doi.org/10.1007/978-3-663-05925-7.

A parametrization

A parametrization of the solution can be written by solving (4) for y to obtain $y = y'x - x^2\sqrt{(y')^2+1} = px - x^2\sqrt{p^2+1}$, where $y' = p \geq 0$. By noting that $dy = pdx$ and $dy = pdx + xdp - 2x\sqrt{p^2+1}dx - x^2\frac{p}{\sqrt{p^2+1}}dp$, a linear differential equation $\frac{dx}{dp} - \frac{xp}{2\sqrt{p^2+1}} = \frac{1}{2\sqrt{p^2+1}}$ is obtained. By solving this equation, we obtain the parametrization

$$\begin{cases} x(p) = \frac{1}{\sqrt[4]{p^2+1}} \int_0^p \frac{dt}{2\sqrt[4]{t^2+1}}, \\ y(p) = px(p) - (x(p))^2\sqrt{p^2+1} \ (p \geq 0). \end{cases} \tag{5}$$

Theorem

Suppose that the tangent line of the curve (4) at the point P, intersects x-axis, y-axis, and the line $x = 1$ at points F, U, and T, respectively. Then

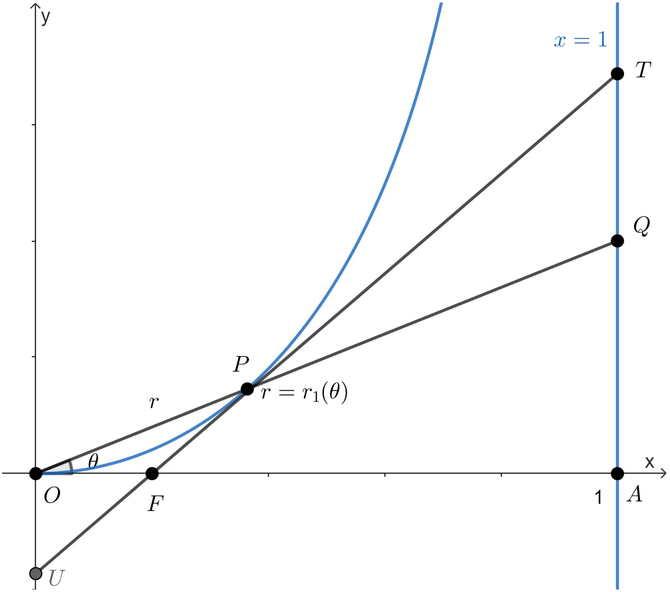
1. $|UP| = |OU| + |TQ|$,
2. $(1 - x) \cdot |UP| = |TQ|$,
3. $x \cdot |PT| = |TQ|$,
4. $\sin \angle QPT = |OP| \cdot \sin^2 \angle TQP$,

where x is the abscissa of the point $P(x, y)$.

Theorem

The radius of the circle through O and tangent to the line UT at the point P is equal to the radius of the circle through O and tangent to the line AT at the point Q.

Plane curve



Theorem

The length of the side PQ, and the difference of lengths of the other two sides of $\triangle PQT$ approach to the same limit B^2 as $x \to 1^-$:

$$\lim_{x\to 1^-} |PQ| = \lim_{x\to 1^-} (|PT| - |TQ|) = \frac{\Gamma\left(\frac{3}{4}\right)^4}{2\pi} = B^2 \approx 0.3588850048,$$

where B is the second lemniscate constant.

Suppose now that P starts to move from O with constant speed along such a curve that the tangent line of the curve at P always passes through Q, which moves as before with the same constant speed from A upwards, along the line $x = 1$. The obtained curve is known as *Pursuit Curve* and defined by the differential equation $\sqrt{1 + (y')^2} = (1 - x)y''$, with initial conditions $y(0) = y'(0) = 0$. The solution found by P. Bouguer in 1732:

$$y = \frac{1}{4}(1 - x)^2 - \frac{1}{2}\ln(1 - x) - \frac{1}{4}.$$

One can check that for this curve $|PQ| = \frac{1}{2} + \frac{(1-x)^2}{2}$, and, therefore, $\lim_{x \to 1^-} |PQ| = \frac{1}{2} > B^2$.

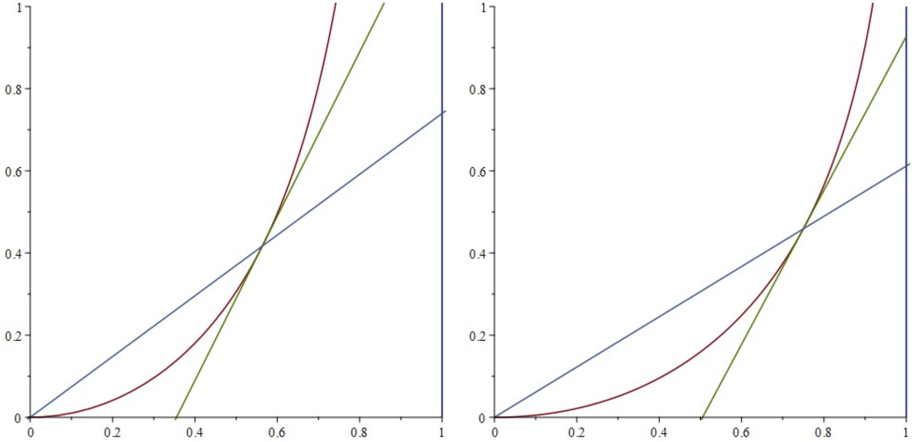


Figure: Comparison of Interception (left, red) and Pursuit (right, red) Curves. Created using parametrization (5), and Bouguer's formula. The tangent lines (green), the lines containing the position vectors (thin blue) of the curves, and the lines $x = 1$ (thick blue) are also shown.

Question 2. Suppose that two points P and Q, initially at $B(0,0,1)$ and $A(1,0,0)$, respectively, move with constant and equal velocity so that Q is on the great circle $z = 0$, $x^2 + y^2 = 1$ of sphere $x^2 + y^2 + z^2 = 1$ with center $O(0,0,0)$, and P is on the great circle through B and Q of the sphere. What curve is defined by the point P?

Answer. We can use spherical coordinates to describe the curve.

$$\phi = 2\tan^{-1} e^{\theta} - \frac{\pi}{2}. \tag{6}$$

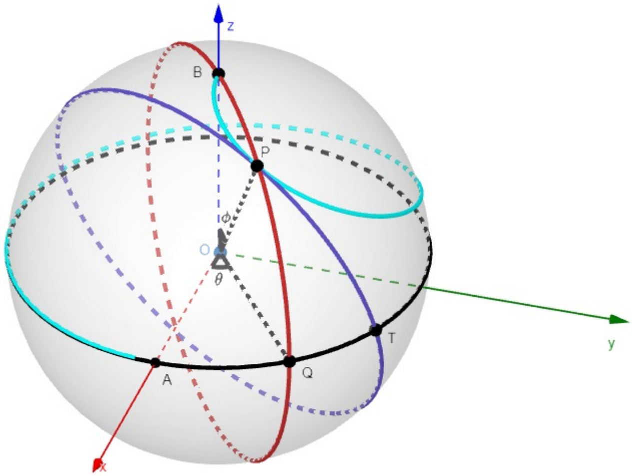


Figure: Interception Curve (light blue) on a Unit Sphere. Its tangent great circle (dark blue), meridian (red) and equator (black) great circles are also shown.

Theorem

$\lim_{\theta\to\infty} |PQ| = 0.$

Proof.

One can observe that

$$\lim_{\theta\to\infty} \phi(\theta) = \lim_{\theta\to\infty} \left(2\tan^{-1} e^{\theta} - \frac{\pi}{2}\right) = 2 \cdot \frac{\pi}{2} - \frac{\pi}{2} = \frac{\pi}{2},$$

and, therefore, the distance between the points $P(1, \theta, \phi)$ and $Q(1, \theta, \frac{\pi}{2})$ approaches zero as the curve winds around the sphere. □

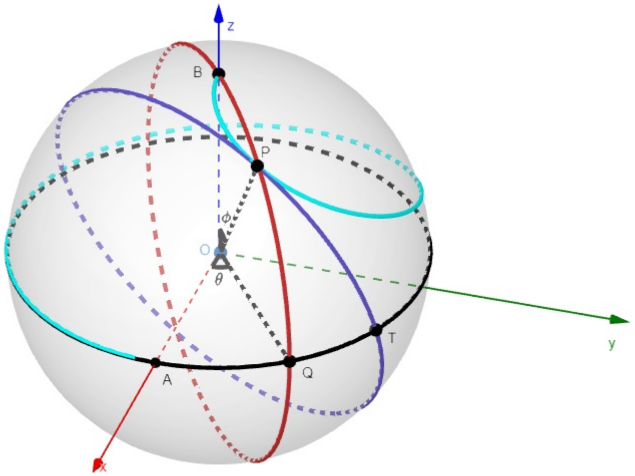


Figure: Interception Curve (light blue) on a Unit Sphere. Its tangent great circle (dark blue), meridian (red) and equator (black) great circles are also shown.

Theorem

If a great circle tangent to the curve $\phi = 2\tan^{-1} e^{\theta} - \frac{\pi}{2}$ *at point* P *of unit sphere intersects the great circle on the plane* xOy *at point* T*, then*

1. *the sum of the lengths of the arcs* PT *and* TQ *is not dependent on* θ*, and* $\widehat{PT} + \widehat{TQ} = \frac{\pi}{2}$*,*
2. *as* θ *increases,* $\widehat{TQ}$ *increases,* $\widehat{PT}$ *decreases, and both approach to* $\frac{\pi}{4}$*, as* $\theta \to \infty$*.*
3. *spherical angle* $\angle QPT$ *is equal to* $\widehat{BP}$*,*
4. *spherical angle* $\angle BPT$ *is equal to* $\widehat{PQ} + \frac{\pi}{2}$*.*

Theorem

If a small circle of the unit sphere passes through point B *and is tangent to the curve (6) at point* P*, then its spherical radius* R *satisfies* $\tan^2 R = \frac{1}{4}\sec^4 \frac{1}{2}\widehat{BP}$*.*

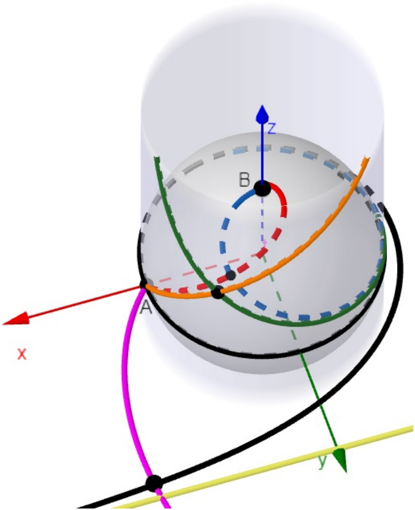


Figure: Mercator and Stereographic projections of the Spherical Interception Curve (blue) and Spherical Spiral (red).

Table: Corresponding curves on the sphere, on the cylinder and on the plane.

Sphere $x^2 + y^2 + z^2 = 1$	
$\phi = 2\tan^{-1} e^\theta - \frac{\pi}{2}$	Interception Curve (blue)
$\left(\frac{\cos\theta}{\cosh\theta}, \frac{\sin\theta}{\cosh\theta}, \tanh\theta\right)$	
Cylinder $x^2 + y^2 = 1$	**(Mercator Projection)**
$y = \ln\coth\frac{x}{2}$	(green)
$y = x$	Helix (orange)
Plane $z = 0$	**(Stereographic Projection)**
$r = \coth\frac{\theta}{2}$	(black)
$r = e^\theta$	Logarithmic Spiral (purple)

Lemma

On a unit sphere with center O, a great circle and one of its poles B is drawn. Two great circles through B intersect the first great circle at Q_1 and Q_2, so that $\widehat{Q_1Q_2} < \frac{\pi}{2}$. Another great circle intersects arcs BQ_1 and BQ_2 at points P_1 and P_2 so that $\widehat{P_1P_2} = \widehat{Q_1Q_2}$. This great circle also intersects the first great circle at T. Then

$$\widehat{P_1T} + \widehat{TQ_2} = \widehat{P_2T} + \widehat{TQ_1} = \frac{\pi}{2}.$$

Lemma

Under conditions of the previous Lemma, let a small circle of the unit sphere be circumscribed about $\triangle BP_1P_2$. If radius of the small circle is R, then

$$\tan^2 R = \frac{1}{4}\sec^4\frac{1}{2}\widehat{P_1P_2}\sec^4\frac{1}{2}\widehat{BP_1}\sec^4\frac{1}{2}\widehat{BP_2}.$$

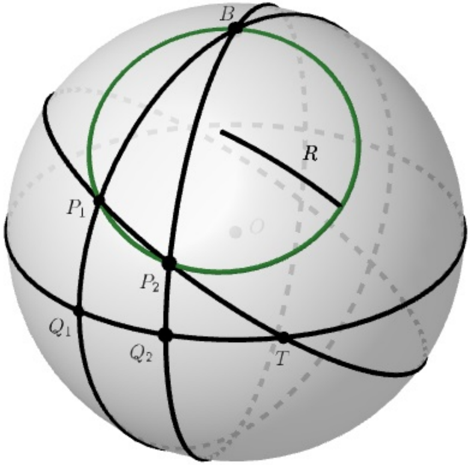


Figure: Lemmata for Spherical Case.

Lemma

A line passing through a point O and points Q_1 and Q_2 on a parallel line are drawn. A line intersects these lines at U and T, and the segments OQ_1 and OQ_2 at points P_1 and P_2, respectively, so that $|P_1P_2| = |Q_1Q_2|$. Then

1. $|OU| + |TQ_2| = |UP_1|$ *and* $|OU| + |TQ_1| = |UP_2|$,
2. *the radii of circles through the points* O, P_1, P_2 *and* O, Q_1, Q_2 *are equal,*
3. *if the distance between the parallel lines is 1, and the distances from the points* P_1 *and* P_2 *to the line* OU *are* x_1 *and* x_2*, respectively, then*

$$\sin\angle Q_1P_1T \cdot \sin\angle Q_2P_2T = |OP_1| \cdot |OP_2| \cdot \sin^2\angle TQ_1P_1 \cdot \sin^2\angle TQ_2P_2,$$

$$x_1x_2 \cdot |P_1T| \cdot |P_2T| = |TQ_1| \cdot |TQ_2|,$$

$$(1 - x_1)(1 - x_2) \cdot |UP_1| \cdot |UP_2| = |TQ_1| \cdot |TQ_2|.$$

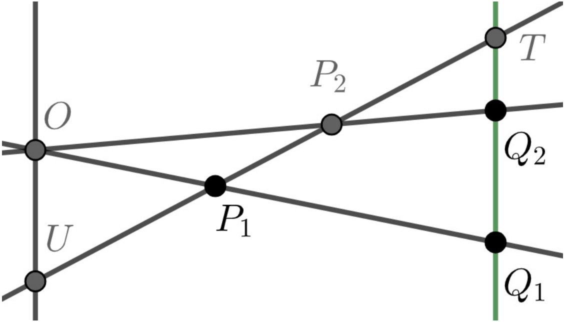


Figure: Lemma for Planar Case.

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