Surfaces with zero mean curvature vector in 4-dimensional spaces

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(N, h): an oriented Riemannian 4-dimensional manifold.

We have a bundle decomposition $\bigwedge^2 TN = \bigwedge^2_+ TN \oplus \bigwedge^2_- TN$.

The twistor spaces associated with N are the sphere bundles in $\bigwedge^2_\pm TN$:

$$U\left(\bigwedge^2_\pm TN\right) := \left\{\Theta \in \bigwedge^2_\pm TN \mid \hat{h}(\Theta, \Theta) = 1\right\}.$$

M: a Riemann surface,

$F : M \longrightarrow N$: a conformal and minimal immersion of M into N,

$\Theta_{F,\pm}$: sections of $U\left(\bigwedge^2_\pm F^*TN\right)$ defined by $\Theta_{F,\pm} := \dfrac{1}{\sqrt{2}}(\xi_1 \wedge \xi_2 \pm \xi_3 \wedge \xi_4)$,

where ξ_1, ξ_2, ξ_3, ξ_4 form a local orthonormal frame field of F^*TN s.t.

- $(\xi_1, \xi_2, \xi_3, \xi_4)$ gives the orientation of N,
- $\xi_1, \xi_2 \in dF(TM)$ so that (ξ_1, ξ_2) gives the orientation of M.

σ: the second fundamental form of F,

$w = u + \sqrt{-1}v$: a local complex coordinate of M,

$$\sigma_{ww} := \sigma\left(\frac{\partial}{\partial w}, \frac{\partial}{\partial w}\right), \quad \frac{\partial}{\partial w} = \frac{1}{2}\left(\frac{\partial}{\partial u} - \sqrt{-1}\frac{\partial}{\partial v}\right).$$

Then $Q := h(\sigma_{ww}, \sigma_{ww})dw^4$ does not depend on the choice of a local complex coordinate w and we can define a complex quartic differential Q on M.

If N is a 4-dimensional Riemannian space form,

then it follows from the equations of Codazzi that Q is holomorphic.

Fact 1 *The following are mutually equivalent:*

(a) *at each point of M, principal curvatures do not depend on the choice of a unit normal vector of F;*

(b) $h(\sigma(T_1, T_1), \sigma(T_1, T_1)) = h(\sigma(T_1, T_2), \sigma(T_1, T_2)),\ \ h(\sigma(T_1, T_1), \sigma(T_1, T_2)) = 0;$

(c) $Q \equiv 0;$

(d) *one of $\Theta_{F,\pm}$ is horizontal w.r.t. the connection $\hat{\nabla}$ of $\bigwedge^2 F^*TN$ induced by the Levi-Civita connection ∇ of h;*

(e) *one of the complex structures $I_{F,\pm}$ corresponding to $\Theta_{F,\pm}$ is parallel w.r.t. ∇;*

(f) *we have one of $I_{F,\pm}\sigma(T_1, T_1) = \sigma(T_1, T_2)$,*

where $T_1 := dF\left(\dfrac{\partial}{\partial u}\right),\, T_2 := dF\left(\dfrac{\partial}{\partial v}\right).$

We say that a minimal immersion F is *isotropic* if F satisfies one of (a) $\sim$ (f) in Fact 1. We see

- (a), (b), (c) and (f) are mutually equivalent,
- (d) and (e) are equivalent.

In addition, (a) and (d) are equivalent ([6]).

Suppose $N = E^4$. Then $U\left(\bigwedge^2_{\pm} TE^4\right) \cong E^4 \times \mathbb{C}P^1$, and an isotropic minimal surface is given by the composition of an isometry of E^4 and a holomorphic immersion into $E^4 = \mathbb{C}^2$.

Suppose $N = S^4$. Then $U\left(\bigwedge^2_{\pm} TS^4\right) = SO(5)/U(2) \cong Sp(2)/U(2) = \mathbb{C}P^3$, and an isotropic minimal surface (superminimal surface) is given by the composition of the twistor map and a holomorphic and horizontal immersion into $\mathbb{C}P^3$ ([4]).

(N,h): an oriented neutral 4-dimensional manifold.

$\Longrightarrow$ The metric h induces an indefinite metric $\hat{h}$ of $\bigwedge^2 TN$ with signature $(2,4)$.

We have a bundle decomposition $\bigwedge^2 TN = \bigwedge^2_+ TN \oplus \bigwedge^2_- TN$ and we see

- $\bigwedge^2_+ TN \perp \bigwedge^2_- TN$ w.r.t. $\hat{h}$,
- the restriction of $\hat{h}$ on each of $\bigwedge^2_\pm TN$ has signature $(1,2)$.

The space-like (or hyperbolic) twistor spaces associated with N are fiber bundles in $\bigwedge^2_\pm TN$ given by

$$U_+\left(\bigwedge\nolimits^2_\pm TN\right) := \left\{\Theta \in \bigwedge\nolimits^2_\pm TN \mid \hat{h}(\Theta,\Theta) = 1\right\}.$$

M: a Riemann surface,

$F : M \longrightarrow N$: a space-like and conformal immersion of M into N
with zero mean curvature vector,

$\Theta_{F,\pm}$: sections of $U_+\left(\bigwedge^2_\pm F^*TN\right)$ defined by $\Theta_{F,\pm} := \dfrac{1}{\sqrt{2}}(\xi_1 \wedge \xi_2 \mp \xi_3 \wedge \xi_4)$,
where ξ_1, ξ_2, ξ_3, ξ_4 form a local pseudo-orthonormal frame field of F^*TN s.t.

- $(\xi_1, \xi_2, \xi_3, \xi_4)$ gives the orientation of N,
- $\xi_1, \xi_2 \in dF(TM)$ so that (ξ_1, ξ_2) gives the orientation of M.

Fact 2 *The following are mutually equivalent:*

(a) *at each point of M, principal curvatures do not depend on the choice of a normal vector $e^{\perp}$ of F with $h(e^{\perp}, e^{\perp}) = -1$;*

(b) $h(\sigma(T_1, T_1), \sigma(T_1, T_1)) = h(\sigma(T_1, T_2), \sigma(T_1, T_2)), \; h(\sigma(T_1, T_1), \sigma(T_1, T_2)) = 0$;

(c) $Q = h(\sigma_{ww}, \sigma_{ww})dw^4$ *vanishes;*

(d) *one of $\Theta_{F,\pm}$ is horizontal w.r.t. $\hat{\nabla}$;*

(e) *one of the complex structures $I_{F,\pm}$ corresponding to $\Theta_{F,\pm}$ is parallel w.r.t. ∇;*

(f) *we have one of $I_{F,\pm}\sigma(T_1, T_1) = \sigma(T_1, T_2)$.*

We say that F is *isotropic* if F satisfies one of (a) $\sim$ (f) in Fact 2.

The time-like twistor spaces associated with N are fiber bundles in $\bigwedge^2_\pm TN$ given by

$$U_-\left(\bigwedge^2_\pm TN\right) := \left\{\Theta \in \bigwedge^2_\pm TN \;\middle|\; \hat{h}(\Theta, \Theta) = -1\right\}.$$

M: a Lorentz surface (two-dimensional manifold with a holomorphic system of paracomplex coordinate neighborhoods),

$F : M \longrightarrow N$: a time-like and conformal immersion of M into N with zero mean curvature vector,

$\Theta_{F,\pm}$: sections of $U_-\left(\bigwedge^2_\pm F^*TN\right)$ defined by $\Theta_{F,\pm} := \dfrac{1}{\sqrt{2}}(\xi_1 \wedge \xi_3 \pm \xi_4 \wedge \xi_2)$,

where ξ_1, ξ_2, ξ_3, ξ_4 form a local pseudo-orthonormal frame field of F^*TN (we suppose that ξ_1, ξ_2 are space-like) s.t.

- $(\xi_1, \xi_2, \xi_3, \xi_4)$ gives the orientation of N,
- $\xi_1, \xi_3 \in dF(TM)$ so that (ξ_1, ξ_3) gives the orientation of M.

Fact 3-1 *The following are equivalent*:

(b) $h(\sigma(T_1, T_1), \sigma(T_1, T_1)) = -h(\sigma(T_1, T_2), \sigma(T_1, T_2)),\ \ h(\sigma(T_1, T_1), \sigma(T_1, T_2)) = 0$;

(c) $Q = h(\sigma_{ww}, \sigma_{ww})dw^4$ *vanishes.*

We say that F is *isotropic* if F satisfies one of (b), (c) in Fact 3-1.

Fact 3-2 *The following are mutually equivalent*:

(d) *one of* $\Theta_{F,\pm}$ *is horizontal w.r.t.* $\hat{\nabla}$;

(e) *one of the paracomplex structures* $J_{F,\pm}$ *corresponding to* $\Theta_{F,\pm}$ *is parallel w.r.t.* ∇;

(f) *we have one of* $J_{F,\pm}\sigma(T_1, T_1) = \sigma(T_1, T_2)$.

In addition, if F *satisfies one of* (d), (e), (f), *then* F *is isotropic.*

We say that F is *strictly isotropic* if F satisfies one of (d), (e), (f) in Fact 3-2 for the orientation of N.

It is possible that although F is isotropic, the covariant derivatives of $\Theta_{F,\pm}$ are not zero but light-like.

Proposition ([2])

Both $\hat{\nabla}\Theta_{F,+}$ *and* $\hat{\nabla}\Theta_{F,-}$ *are light-like or zero if and only if* F *satisfies one of the following*:

(a) *the shape operator of a light-like normal vector field vanishes, and then* Q *vanishes*;

(b) *the shape operator of any normal vector field is zero or light-like, and then* Q *is null or zero.*

Suppose that N is a 4-dimensional neutral space form.

Condition (a) implies that a light-like normal vector field of the surface is contained in a constant direction.
The conformal Gauss map of a time-like surface in a 3-dimensional Lorentzian space form of Willmore type with $Q \equiv 0$ satisfies Condition (a) ([1]).

By the equations of Gauss-Codazzi-Ricci, we obtain

Theorem ([2])
$F : M \longrightarrow N$: *a time-like and conformal immersion of M into N with zero mean curvature vector.*
Then F satisfies Condition (b) *if and only if the following hold:*

- *the curvature K of the induced metric by F is identically equal to the constant sectional curvature L_0 of N;*
- *the second fundamental form σ of F satisfies a "light-like condition";*
- *the curvature tensor of the normal connection $\nabla^{\perp}$ of F vanishes.*

Remark The Gauss-Weingarten equations for surfaces as in the above theorem are given by

$$\tilde{\nabla}_{T_1}(T_1\ T_2\ N_1\ N_2\ F) = (T_1\ T_2\ N_1\ N_2\ F)A,$$
$$\tilde{\nabla}_{T_2}(T_1\ T_2\ N_1\ N_2\ F) = (T_1\ T_2\ N_1\ N_2\ F)B,$$

where

$$A = \begin{bmatrix} \lambda_u & \lambda_v & -\alpha_+ & \alpha_- & 1 \\ \lambda_v & \lambda_u & \varepsilon\alpha_+ & -\varepsilon\alpha_- & 0 \\ \alpha_+ & \varepsilon\alpha_+ & \lambda_u & \gamma_u & 0 \\ \alpha_- & \varepsilon\alpha_- & \gamma_u & \lambda_u & 0 \\ -L_0e^{2\lambda} & 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} \lambda_v & \lambda_u & -\varepsilon\alpha_+ & \varepsilon\alpha_- & 0 \\ \lambda_u & \lambda_v & \alpha_+ & -\alpha_- & 1 \\ \varepsilon\alpha_+ & \alpha_+ & \lambda_v & \gamma_v & 0 \\ \varepsilon\alpha_- & \alpha_- & \gamma_v & \lambda_v & 0 \\ 0 & L_0e^{2\lambda} & 0 & 0 & 0 \end{bmatrix}$$

and

- $\varepsilon = 1$ or -1,
- $\lambda_{uu} - \lambda_{vv} = -L_0e^{2\lambda}$,
- $\alpha_\pm = \pm\dfrac{1}{2e^\lambda}(\phi(u+\varepsilon v)e^\gamma \pm \psi(u+\varepsilon v)e^{-\gamma})$,
- γ is a function of u, v, and ϕ, ψ are functions of one variable.

THANK YOU VERY MUCH

FOR YOUR ATTENTION!