Surfaces with zero mean curvature vector in 4-dimensional spaces

Naoya Ando (Kumamoto University, Japan)

International Scientific Online Conference
Algebraic and Geometric Methods of Analysis 2023
\((N, h)\): an oriented Riemannian 4-dimensional manifold.

We have a bundle decomposition \(\Lambda^2 TN = \Lambda^2_+ TN \oplus \Lambda^2_- TN\).

The twistor spaces associated with \(N\) are the sphere bundles in \(\Lambda^2_\pm TN\):

\[
U\left(\Lambda^2_\pm TN\right) := \{\Theta \in \Lambda^2_\pm TN \mid \hat{h}(\Theta, \Theta) = 1\}.
\]

\(M\): a Riemann surface,

\(F : M \to N\): a conformal and minimal immersion of \(M\) into \(N\),

\(\Theta_{F,\pm}\): sections of \(U\left(\Lambda^2_\pm F^*TN\right)\) defined by \(\Theta_{F,\pm} := \frac{1}{\sqrt{2}}(\xi_1 \wedge \xi_2 \pm \xi_3 \wedge \xi_4)\),

where \(\xi_1, \xi_2, \xi_3, \xi_4\) form a local orthonormal frame field of \(F^*TN\) s.t.

- \((\xi_1, \xi_2, \xi_3, \xi_4)\) gives the orientation of \(N\),
- \(\xi_1, \xi_2 \in dF(TM)\) so that \((\xi_1, \xi_2)\) gives the orientation of \(M\).
\( \sigma \): the second fundamental form of \( F \),
\( w = u + \sqrt{-1}v \): a local complex coordinate of \( M \),
\[
\sigma_{ww} := \sigma \left( \frac{\partial}{\partial w}, \frac{\partial}{\partial w} \right), \quad \frac{\partial}{\partial w} = \frac{1}{2} \left( \frac{\partial}{\partial u} - \sqrt{-1}\frac{\partial}{\partial v} \right).
\]

Then \( Q := h(\sigma_{ww}, \sigma_{ww})dw^4 \) does not depend on the choice of a local complex coordinate \( w \) and we can define a complex quartic differential \( Q \) on \( M \).

If \( N \) is a 4-dimensional Riemannian space form, then it follows from the equations of Codazzi that \( Q \) is holomorphic.
Fact 1 The following are mutually equivalent:

(a) at each point of $M$, principal curvatures do not depend on the choice of a unit normal vector of $F$;

(b) $h(\sigma(T_1, T_1), \sigma(T_1, T_1)) = h(\sigma(T_1, T_2), \sigma(T_1, T_2)), \quad h(\sigma(T_1, T_1), \sigma(T_1, T_2)) = 0$;

(c) $Q \equiv 0$;

(d) one of $\Theta_{F,\pm}$ is horizontal w.r.t. the connection $\hat{\nabla}$ of $\wedge^2 F^*TN$ induced by the Levi-Civita connection $\nabla$ of $h$;

(e) one of the complex structures $I_{F,\pm}$ corresponding to $\Theta_{F,\pm}$ is parallel w.r.t. $\nabla$;

(f) we have one of $I_{F,\pm}\sigma(T_1, T_1) = \sigma(T_1, T_2),$

where $T_1 := dF \left( \frac{\partial}{\partial u} \right), T_2 := dF \left( \frac{\partial}{\partial v} \right)$. 
We say that a minimal immersion $F$ is *isotropic* if $F$ satisfies one of (a) $\sim$ (f) in Fact 1. We see

- (a), (b), (c) and (f) are mutually equivalent,
- (d) and (e) are equivalent.

In addition, (a) and (d) are equivalent ([6]).

Suppose $N = E^4$. Then $U\left(\bigwedge_{\pm}^2 TE^4\right) \cong E^4 \times \mathbb{C}P^1$, and an isotropic minimal surface is given by the composition of an isometry of $E^4$ and a holomorphic immersion into $E^4 = \mathbb{C}^2$.

Suppose $N = S^4$. Then $U\left(\bigwedge_{\pm}^2 TS^4\right) = SO(5)/U(2) \cong Sp(2)/U(2) = \mathbb{C}P^3$, and an isotropic minimal surface (superminimal surface) is given by the composition of the twistor map and a holomorphic and horizontal immersion into $\mathbb{C}P^3$ ([4]).
\((N, h)\): an oriented neutral 4-dimensional manifold.

\[\Rightarrow\] The metric \(h\) induces an indefinite metric \(\hat{h}\) of \(\bigwedge^2 TN\) with signature \((2, 4)\).

We have a bundle decomposition \(\bigwedge^2 TN = \bigwedge^2_+ TN \oplus \bigwedge^2_- TN\) and we see

- \(\bigwedge^2_+ TN \perp \bigwedge^2_- TN\) w.r.t. \(\hat{h}\),
- the restriction of \(\hat{h}\) on each of \(\bigwedge^2_\pm TN\) has signature \((1, 2)\).

The space-like (or hyperbolic) twistor spaces associated with \(N\) are fiber bundles in \(\bigwedge^2 TN\) given by

\[U_+ \left( \bigwedge^2 TN \right) := \left\{ \Theta \in \bigwedge^2 TN \mid \hat{h}(\Theta, \Theta) = 1 \right\}.\]
$M$: a Riemann surface,

$F : M \rightarrow N$: a space-like and conformal immersion of $M$ into $N$

with zero mean curvature vector,

$\Theta_{F,\pm}$: sections of $U_+\left(\bigwedge^2_{\pm} F^*TN\right)$ defined by $\Theta_{F,\pm} := \frac{1}{\sqrt{2}}(\xi_1 \wedge \xi_2 \mp \xi_3 \wedge \xi_4)$,

where $\xi_1, \xi_2, \xi_3, \xi_4$ form a local pseudo-orthonormal frame field of $F^*TN$ s.t.

- $(\xi_1, \xi_2, \xi_3, \xi_4)$ gives the orientation of $N$,

- $\xi_1, \xi_2 \in dF(TM)$ so that $(\xi_1, \xi_2)$ gives the orientation of $M$. 
Fact 2 The following are mutually equivalent:

(a) at each point of $M$, principal curvatures do not depend on the choice of a normal vector $e^\perp$ of $F$ with $h(e^\perp, e^\perp) = -1$;

(b) $h(\sigma(T_1, T_1), \sigma(T_1, T_1)) = h(\sigma(T_1, T_2), \sigma(T_1, T_2))$, $h(\sigma(T_1, T_1), \sigma(T_1, T_2)) = 0$;

(c) $Q = h(\sigma_{ww}, \sigma_{ww})dw^4$ vanishes;

(d) one of $\Theta_{F,\pm}$ is horizontal w.r.t. $\hat{\nabla}$;

(e) one of the complex structures $I_{F,\pm}$ corresponding to $\Theta_{F,\pm}$ is parallel w.r.t. $\nabla$;

(f) we have one of $I_{F,\pm}\sigma(T_1, T_1) = \sigma(T_1, T_2)$.

We say that $F$ is isotropic if $F$ satisfies one of (a) $\sim$ (f) in Fact 2.
The time-like twistor spaces associated with $N$ are fiber bundles in $\bigwedge^2_{\pm}TN$ given by

$$U_-(\bigwedge^2_{\pm}TN) := \left\{ \Theta \in \bigwedge^2_{\pm}TN \mid \hat{h}(\Theta, \Theta) = -1 \right\}.$$ 

$M$: a Lorentz surface (two-dimensional manifold with a holomorphic system of paracomplex coordinate neighborhoods),

$F: M \longrightarrow N$: a time-like and conformal immersion of $M$ into $N$ with zero mean curvature vector,

$\Theta_{F,\pm}$: sections of $U_-(\bigwedge^2_{\pm}F^*TN)$ defined by $\Theta_{F,\pm} := \frac{1}{\sqrt{2}}(\xi_1 \wedge \xi_3 \pm \xi_4 \wedge \xi_2)$, where $\xi_1, \xi_2, \xi_3, \xi_4$ form a local pseudo-orthonormal frame field of $F^*TN$ (we suppose that $\xi_1, \xi_2$ are space-like) s.t.

- $(\xi_1, \xi_2, \xi_3, \xi_4)$ gives the orientation of $N$,
- $\xi_1, \xi_3 \in dF(TM)$ so that $(\xi_1, \xi_3)$ gives the orientation of $M$. 
Fact 3-1  The following are equivalent:

(b) \( h(\sigma(T_1, T_1), \sigma(T_1, T_1)) = -h(\sigma(T_1, T_2), \sigma(T_1, T_2)), \ h(\sigma(T_1, T_1), \sigma(T_1, T_2)) = 0; \)

(c) \( Q = h(\sigma_{ww}, \sigma_{ww})dw^4 \) vanishes.

We say that \( F \) is \textit{isotropic} if \( F \) satisfies one of (b), (c) in Fact 3-1.

Fact 3-2  The following are mutually equivalent:

(d) one of \( \Theta_{F,\pm} \) is horizontal w.r.t. \( \hat{\nabla} \);

(e) one of the paracomplex structures \( J_{F,\pm} \) corresponding to \( \Theta_{F,\pm} \) is parallel w.r.t. \( \nabla \);

(f) we have one of \( J_{F,\pm}\sigma(T_1, T_1) = \sigma(T_1, T_2) \).

In addition, if \( F \) satisfies one of (d), (e), (f), then \( F \) is \textit{isotropic}.

We say that \( F \) is \textit{strictly isotropic} if \( F \) satisfies one of (d), (e), (f) in Fact 3-2 for the orientation of \( N \).
It is possible that although $F$ is isotropic, the covariant derivatives of $\Theta_{F,\pm}$ are not zero but light-like.

**Proposition ([2])**

Both $\hat{\nabla}\Theta_{F,+}$ and $\hat{\nabla}\Theta_{F,-}$ are light-like or zero if and only if $F$ satisfies one of the following:

(a) the shape operator of a light-like normal vector field vanishes, and then $Q$ vanishes;

(b) the shape operator of any normal vector field is zero or light-like, and then $Q$ is null or zero.
Suppose that $N$ is a 4-dimensional neutral space form.

Condition (a) implies that a light-like normal vector field of the surface is contained in a constant direction.

The conformal Gauss map of a time-like surface in a 3-dimensional Lorentzian space form of Willmore type with $Q \equiv 0$ satisfies Condition (a) ([1]).

By the equations of Gauss-Codazzi-Ricci, we obtain

**Theorem ([2])**

$F : M \rightarrow N$: a time-like and conformal immersion of $M$ into $N$

with zero mean curvature vector.

Then $F$ satisfies Condition (b) if and only if the following hold:

- the curvature $K$ of the induced metric by $F$ is identically equal to the constant sectional curvature $L_0$ of $N$;
- the second fundamental form $\sigma$ of $F$ satisfies a “light-like condition”;
- the curvature tensor of the normal connection $\nabla^\perp$ of $F$ vanishes.
**Remark** The Gauss-Weingarten equations for surfaces as in the above theorem are given by

\[
\tilde{\nabla}_{T_1}(T_1 \ T_2 \ N_1 \ N_2 \ F) = (T_1 \ T_2 \ N_1 \ N_2 \ F)A,
\]

\[
\tilde{\nabla}_{T_2}(T_1 \ T_2 \ N_1 \ N_2 \ F) = (T_1 \ T_2 \ N_1 \ N_2 \ F)B,
\]

where

\[
A = \begin{bmatrix}
\lambda_u & \lambda_v & -\alpha_+ & \alpha_- & 1 \\
\lambda_v & \lambda_u & \varepsilon\alpha_+ & -\varepsilon\alpha_- & 0 \\
\alpha_+ & \varepsilon\alpha_+ & \lambda_u & \gamma_u & 0 \\
\alpha_- & \varepsilon\alpha_- & \gamma_u & \lambda_u & 0 \\
-L_0e^{2\lambda} & 0 & 0 & 0 & 0
\end{bmatrix}, \quad B = \begin{bmatrix}
\lambda_v & \lambda_u & -\varepsilon\alpha_+ & \varepsilon\alpha_- & 0 \\
\lambda_u & \lambda_v & \alpha_+ & -\alpha_- & 1 \\
\varepsilon\alpha_+ & \alpha_+ & \lambda_v & \gamma_v & 0 \\
\varepsilon\alpha_- & \alpha_- & \gamma_v & \lambda_v & 0 \\
0 & L_0e^{2\lambda} & 0 & 0 & 0
\end{bmatrix}
\]

and

- \( \varepsilon = 1 \) or \(-1\),
- \( \lambda_{uu} - \lambda_{vv} = -L_0e^{2\lambda} \),
- \( \alpha_\pm = \pm \frac{1}{2e^\gamma}(\phi(u + \varepsilon v)e^\gamma \pm \psi(u + \varepsilon v)e^{-\gamma}) \),
- \( \gamma \) is a function of \( u, v \), and \( \phi, \psi \) are functions of one variable.
THANK YOU VERY MUCH
FOR YOUR ATTENTION!