Characterizing Linear Mappings Through Unital Algebra

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Abstract

In this paper, we characterize two linear mappings δ and τ satisfying the identity $x \circ y^* = 0 \Rightarrow 0 = \delta(x) \circ y^* + x \circ \tau(y)^*$ for all $x, y \in A$, where A is an algebra over a real or complex field K from a unital algebra into its unital bimodule.

Introduction

※ The structure of linear mappings behaving like Jordan derivations at commutative zero products has been studied extensively. We refer the researchers to Liu [1] and H. Ghahramani, M. N. Ghosseiri, and L. Heidarizadeh [2] for more details. As is well known, the problem of linear mappings preserving fixed products is a very interesting item in the field of operator algebra. Derivations that can be completely determined by the local action on some subsets of algebra have attracted attention of many researchers. Historically, the study of derivation was initiated during the 1950s and 1960s. Derivations of rings got a tremendous development in 1957, when Posner [3] established two very striking results in the case of prime rings.

⁎ Here, we only focus on derivable mappings at special points. There are a considerable number of influential results on derivable mappings at zero, unit, invertible elements, separating points, idempotent elements and so on. In some results, a derivable mapping at zero is described in terms of a generalized derivation. Jing, Lu and Li [9] described continuous derivable mappings at zero on von Neumann algebras. It was studied in [16] for continuous derivable mappings at zero from unital C^*-algebras into unital Banach bimodules. An and Cai [17] discussed continuous derivable mappings at arbitrary but fixed products on von Neumann algebras. In [11], Lu considered continuous derivable mappings from a unital Banach algebra into its unital Banach bimodule at left or right separating points. Furthermore, the authors Jiankui Li and Zhou [15] generalized this result without the assumption of continuity.
Moreover, Jiankui Li, Shan Li, and Kaijia Luo [13, 2023] they gave a complete description of linear mappings δ and τ from G into M satisfying
$\delta(A)B^* + A\tau(B)^* = 0$ for any $A, B \in G$ with $AB^* = 0$ or
$\delta(A) \circ B^* + A \circ \tau(B)^* = 0$ for any $A, B \in G$ with $A \circ B^* = 0$, where
$AB = AB + BA$ is the Jordan product.

Preliminaries

♣ Throughout this paper, let A be an associative algebra over the complex field . Some papers have the structure of derivable mappings at zero Jordan products. Jordan product is denoted by " $\circ$ ": $A \circ B = AB + BA$. A linear mapping δ from G into M is a Jordan derivable mapping at zero Jordan products if $\delta(A) \circ B + A \circ \delta(B) = 0$ for each A, B in G with $A \circ B = 0$. Let us fix some more notations. A linear mapping δ from G into M is called a derivation if $\delta(AB) = \delta(A)B + A\delta(B)$ for each $A, B \in G$; δ is called a Jordan derivation if $\delta(A^2) = \delta(A)A + A\delta(A)$ for each $A \in G$, which is equivalent to $\delta(A \circ B) = \delta(A) \circ B + A \circ \delta(B)$ for each A, B in G. It is clear that every derivation is a Jordan derivation. But the reverse is not always true. Besides, a linear mapping δ from G into M is called a Jordan left derivation if $\delta(A^2) = 2A\delta(A)$ for each $A \in G$. An element W in G is a left (right) separating point of M if $WM = 0$ (or $MW = 0$) implies $M = 0$ for each $M \in M$. W is called a separating point if W is both a left separating point and a right separating point. It is easy to see that left (right) invertible elements in G are left (right) separating points of M, and invertible elements in G are separating points of M.

♣ Based on [14], we denote by $T(G)$ the subalgebra of G generated by all idempotents in G. Let G be an algebra. An A-bimodule M is said to have the property $\diamond$, if there is an ideal $J \subseteq (G)$ of A such that $\{m \in M : xmx = 0$ for every $x \in J\} = 0$. Also, we need to the definition of weak inverse element. In the theory of semigroups, a weak inverse of an element x in a semigroup $(S, \cdot)$ is an element y such that $y \cdot x \cdot y = y$. If every element has a weak inverse, the semigroup is called an E-inversive or E-dense semigroup.

We begin our discussion with the following result.

Lemma (14, Theorem 3.3)

If θ is a bilinear mapping from $G \times G$ into a vector spaces χ such that $A, B \in G, A \circ B = 0 = \theta(A, B) = 0$, then $\theta(A, X) = \frac{1}{2}\theta(AX, I) + \frac{1}{2}\theta(XA, I)$ for every A in G and X in $T(G)$.

Theorem (3.1)

Let A be a unital algebra and M be a unital A-bimodule with the property $\diamond$. Suppose that δ is a linear mapping from A into M satisfying the relation, for all $x, y \in A, x \circ y = 0 \Rightarrow \delta(x) \circ y - x \circ \delta(y) = 0$ and each element of A has a weak inverse. Then A has zero ideal.

Theorem (3.2)

Let A be a unital algebra and M be a unital A-bimodule with the property $\diamond$. Suppose that δ and τ are linear mappings from A into M satisfying the relation, for all $x, y \in A, x \circ y = 0 \Rightarrow \delta(x) \circ y + x \circ \tau(y) = 0$ and $[A, (\delta - \tau)] = 0$. Then there exists a Jordan derivation $\triangle$ from A into M such that $\triangle(x) = 0$ for every x in A.

Theorem (3.3)

Let A be a unital $*$-algebra and M be a unital $*$-A-bimodule with the property $\diamond$. If δ and τ are linear mappings from A into M satisfying the identity, for all $x, y \in A, x \circ y^* = 0 \Rightarrow \delta(x) \circ y^* + x \circ \tau(y)^* = 0$, and A is a separating point of M. Then there exist Jordan derivations $\triangle$ and Γ from A into M and $\delta(A) = 0$.

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Thank you!