Thurston norm and Euler classes of bounded mean curvature foliations on hyperbolic 3-Manifolds

Dmitry V.Bolotov

B. Verkin ILTPE of NASU

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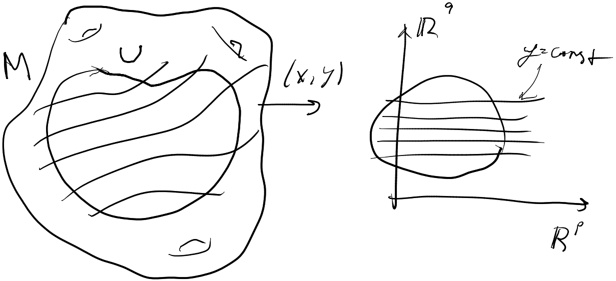
Foliations

A foliation $\mathcal{F}$ of codimension q on smooth manifold M of dimension $n = p + q$ is a partition $\{L_\alpha\}$ of M into connected subsets with the following property. For every point of M there is a neighborhood U and a chart $U \stackrel{(x,y)}{\to} \mathbb{R}^p \times \mathbb{R}^q$ $((x,y) = (x_1, \dots, x_p, y_1, \dots, y_q))$ such that for each leaf L_α are the connected component $U \cap L_\alpha$ defined by the equation $y = const$.



Let M be an oriented Riemannian manifold. Then

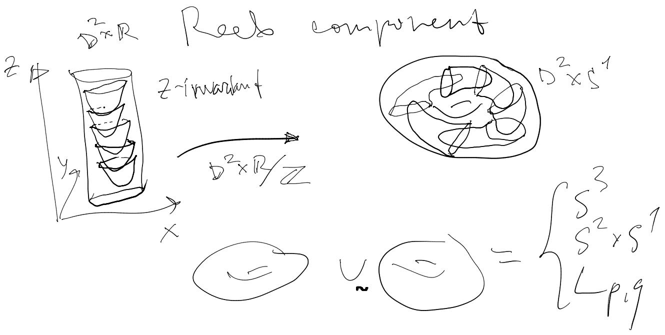
$$TM = T\mathcal{F} \oplus (T\mathcal{F})^{\perp}$$

where $T\mathcal{F}$ is distribution tangent to $\mathcal{F}$. $\mathcal{F}$ is transversely oriented if $(T\mathcal{F})^{\perp}$ is an oriented vector bundle. Note that in this case $T\mathcal{F}$ is also oriented. In particular, the leaves of $\mathcal{F}$ are oriented submanifolds.

If a codimension of $\mathcal{F}$ is equal to one then $T\mathcal{F} = \ker \omega$, where ω is nonsingular 1-form.

Recall that by the Frobenius theorem if ω is nonsingular 1-form, then $\omega \wedge d\omega = 0$ is equivalent to the fact that the distribution $L = \ker \omega$ tangent to a foliation.

Examples



Novikov theorem (1967)

1. For compact, orientable and transversely orientable foliated 3-manifolds $(M, \mathcal{F})$, the following are equivalent.
   a) The foliation $\mathcal{F}$ has a Reeb component.
   b) There is a leaf L of $\mathcal{F}$ that is not π_1-injective. That is, the inclusion $i : L \to M$ induces a homomorphism $i_* : \pi_1(L) \to \pi_1(M)$ with nontrivial kernel.
   c) Some leaf of $\mathcal{F}$ contains a non-trivial vanishing cycle.
2. The leaf supporting the non-trivial vanishing cycle is a torus bounding a Reeb component.

As a corollary Novikov proved that every foliation on a closed oriented 3-manifold M with a finite fundamental group (e.g. $M = S^3$) contains a Reeb component.

Generelized Reeb components

Let (M, g) be a closed oriented three-dimensional Riemannian manifold and let $\mathcal{F}$ be a transversely oriented smooth foliation of codimension one on M.

Definition

A subset A of a three-dimensional compact orientable manifold M with a given transversely orientable foliation $\mathcal{F}$ of codimension one is called a *generalized Reeb component* if A is a connected three-dimensional submanifold of M with a boundary which is a family of compact leaves with the property that any transversal to $\mathcal{F}$ vector field restricted to the boundary ∂A of A is directed either everywhere inwards or everywhere outwards of the Reeb component A. In particular, the Reeb component R is a generalized Reeb component. It is not difficult to show that ∂A is a family of tori (Goodman -1975).

Taut foliations

Recall that a foliation $\mathcal{F}$ is *taut* if a closed transversal passes through each leaf of $\mathcal{F}$. The following properties of $\mathcal{F}$ are equivalent (Sullivan -1979 , Goodman - 1975):

- $\mathcal{F}$ is taut;
- $\mathcal{F}$ is *minimal* (i.e. its leaves are minimal submanifolds of M) for some Riemannian metric on M;
- $\mathcal{F}$ does not contain generalized Reeb components.

Thurston norm

Let M be a closed, oriented 3-manifold. The Thurston norm on $H_2(M, \mathbb{Z})$ is defined as follows (Thurston-1986):

$||a||_{Th} = inf\{\chi_-(\Sigma)|\Sigma$is an emb. surf. representing $a \in H_2(M, \mathbb{Z})\}$,

where $\chi_-(\Sigma) = max\{-\chi(\Sigma), 0\}$. Recall that $\chi(\Sigma) = 2 - 2g$ denotes the Euler characteristic of a surface Σ of genus g. When Σ is not connected, define $\chi_-(\Sigma)$ to be the sum $\chi_-(\Sigma_1) + \cdots + \chi_-(\Sigma_k)$, where $\Sigma_i,\ i = 1, \ldots, k$ are the connected components of Σ. As Thurston showed, the Thurston norm can be extended in a unique way to the seminorm in $H_2(M, \mathbb{R})$.

Dual Thurston norm

Let M be a closed, oriented 3-manifold, and suppose that M contains no non-separating 2 - spheres or tori, for example, M is a closed oriented hyperbolic 3-Manifold, then $|| \cdot ||_{Th}$ is a norm.

Definition

The dual Thurston norm can be defined on $H^2(M, \mathbb{R})$ by the formula

$$||\alpha||^*_{Th} = \sup_{\Sigma} \frac{< \alpha, [\Sigma] >}{2g(\Sigma) - 2}, \qquad (1)$$

where $\alpha \in H^2(M, \mathbb{R})$ and the supremum being taken over all connected, oriented surfaces Σ embedded in M whose genus g is at least 2.

Thurston proved that the convex hull of the Euler classes of taut foliations on M is the unit ball for the dual Thurston norm. In particular, the Thurston norm $||e(\mathcal{F})||^*_{Th}$ of the Euler class $e(\mathcal{F}) \in H^2(M, \mathbb{R})$ of a taut foliation $\mathcal{F}$ no more then one.

Previous result (B. -2022)

Let $V_0 > 0, i_0 > 0, K_0 \geq 0$ be fixed constants, and M be a closed oriented three-dimensional Riemannian manifold with the following properties: the volume $Vol(M) \leq V_0$; the sectional curvature K of M satisfies the inequality $K \leq K_0$; $\min\{inj(M), \frac{\pi}{2\sqrt{K_0}}\} \geq i_0$, where $inj(M)$ is the injectivity radius of of M.

Let us set

$$H_0 = \begin{cases} \min\{\frac{2\sqrt{3}i_0^2}{V_0}, \sqrt[3]{\frac{2\sqrt{3}}{V_0}}\}, & \text{if } K_0 = 0, \\ \min\{\frac{2\sqrt{3}i_0^2}{V_0}, x_0\}, & \text{if } K_0 > 0, \end{cases}$$

where x_0 is the root of the equation

$$\frac{1}{K_0}\operatorname{arccot}^2 \frac{x}{\sqrt{K_0}} - \frac{V_0}{2\sqrt{3}}x = 0.$$

Then any smooth transversely oriented foliation $\mathcal{F}$ of codimension one on M, such that the modulus of the mean curvature of its leaves satisfies the inequality $|H| < H_0$, should be taut, in particular, minimal for some Riemannian metric on M.

Main Theorem

Let M be a closed oriented hyperbolic 3-Manifold and $\mathcal{F}$ be a two-dimensional transversely oriented foliation $\mathcal{F}$ whose leaves have the modulus of mean curvature bounded above by the fixed positive constant H_0. Then

a) If $H_0 \leq 1$, we have $\mathcal{F}$ is taut and $||e(\mathcal{F})||^*_{Th} \leq 1$.

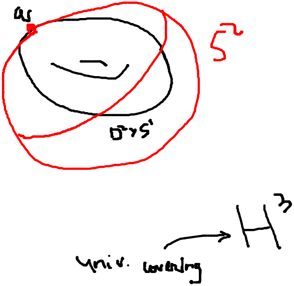
b) If $H_0 > 1$, we have

$$||e(\mathcal{F})||^*_{Th} \leq 2\pi \frac{1600 H_0^2 Vol(M)^2}{C_0^3 inj(M)} + \frac{300 Vol(M)}{inj(M)} + 1,$$

where $C_0 = 2\min\{inj(M), (\coth)^{-1}(H_0)\}$, $Vol(M)$ is the volume of M and $inj(M)$ is the injectivity radius of M.

the proof of p. u)

1)



2)

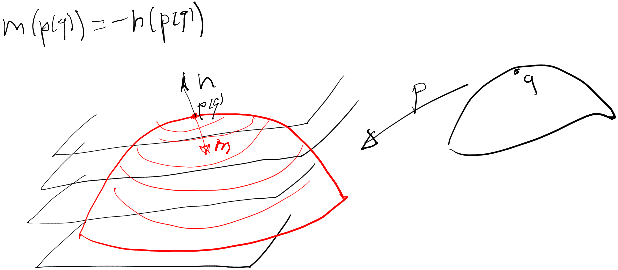
Thurston's approach to calculating $e(T\mathcal{F})$

Let $p : N^2 \to (M, \mathcal{F})$ be a general position immersion of a closed oriented surface N^2. The induced foliation $\mathcal{F}' = p^{-1}(\mathcal{F} \cap p(N^2))$ on N^2 can be oriented outside the singular points. To verify this let us take a normal vector field n to the foliation $\mathcal{F}$ and for all $x = p(z) \in p(N^2)$ consider the orthogonal projection $n'(x)$ of the normal $n(x)$ to $\mathcal{F}$ onto the tangent plane $p_*(T_z(N^2))$, which in the case where z is not a singular point uniquely determines the unit tangent vector e' to the leaf $\mathcal{L}'_z \in \mathcal{F}'$, $z \in \mathcal{L}'_z$, such that the frame $\{e', p_*^{-1} \frac{n'}{|n'|}\}$ defines a positive orientation of $T_z(N^2)$. Now we can define a smooth vector field X on N^2 tangent to $\mathcal{F}'$ whose zeros correspond to the singular points of $\mathcal{F}'$ putting

$$X = |n'|e'. \qquad (2)$$

As W. Thurston showed, to calculate the value of the Euler class $e(T\mathcal{F})$ of the foliation $\mathcal{F}$ on the class $[p, N^2]$, it suffices to calculate the total index of singular points of the vector field X on N^2 taking into account the orientation of $p_*(T_q(N^2))$ at singular points. Since M is oriented we can uniquely choose a unit normal vector $m \in T_{p(q)}M$ to the plane $p_*(T_q(N^2)),\ q \in N^2$, which defines the orientation of $p_*(T_q(N^2))$ coming from the orientation of $T_q(N^2)$.

We say that a singular point $q \in N^2$ is of *negative* type if $m(p(q)) = -n(p(q))$. In the case when $m(p(q)) = n(p(q))$ the type of the singular point is called *positive*.



We denote by I_N the sum of indices of negative type singular points of the vector field X and by I_P the sum of indices of positive type singular points. Then, as W. Thurston showed, the value of the Euler class $e(T\mathcal{F})$ on the homology class $[N^2, p]$ is calculated as follows:

$$e(T\mathcal{F})([N^2, p]) = e(p^*(T\mathcal{F}))([N^2]) = I_P - I_N. \qquad (3)$$

Harmonic maps to the circle

Stern - 2019

Let $u : M \to S^1$ be a harmonic map representing the class $[u] \in [M, S^1] \cong H^1(M;\mathbb{Z}) \overset{PD}{\cong} H_2(M;\mathbb{Z})$. Then

$$2\pi \int_{\theta \in S^1} \chi(\Sigma_{\theta \in S^1}) \geq \frac{1}{2} \int_{\theta \in S^1} \int_{\Sigma_\theta} (|du|^{-2} |Hess(u)|^2 + R_M), \quad (4)$$

where $\Sigma_\theta = u^{-1}\theta$, $\theta \in S^1 = \mathbb{R}/\mathbb{Z}$ is a regular value of u, and R_M is the scalar curvature of M.

Remark that $u^* d\theta$ is a harmonic 1-form on M corresponding to integer class $[u]$ in $H^1(M, \mathbb{R})$.

Jeffrey F. Brock and Nathan M. Dunfield -2017

1) For all closed orientable hyperbolic 3-manifolds M one has

$$\frac{\pi}{\sqrt{Vol(M)}}||\cdot||_{Th} \leq ||\cdot||_{L^2} \leq \frac{10\pi}{\sqrt{inj(M)}}||\cdot||_{Th},$$

This norms are supposed on $H^1(M;\mathbb{R})$.
2) If α is a harmonic 1-form ($d\alpha = \delta\alpha = 0$) on a close hyperbolic 3-manifold then

$$||\alpha||_{L^\infty} \leq \frac{5}{\sqrt{inj(M)}}||\alpha||_{L^2}$$

Recall

$$||\alpha||_{L^2} = \sqrt{\int_M \alpha \wedge *\alpha} = \sqrt{\int_M |\alpha|^2}$$

$$||\alpha||_{L^\infty} = \max_{p\in M} |\alpha_p|$$

Some intermediate results

Theorem

Let M be a closed oriented hyperbolic 3-Manifold and $\mathcal{F}$ be a two-dimensional transversely oriented foliation $\mathcal{F}$ whose leaves have the modulus of mean curvature bounded above by the fixed positive constant H_0. For each harmonic map $u : M \to S^1$ there exists a regular value $\theta \in S^1 = \mathbb{R}/\mathbb{Z}$ such that:

- $\chi_-(\Sigma_\theta) \leq C_1(Vol(M), inj(M), H_0)||[\Sigma_\theta]||_{Th}$;
- *the number of intersection circles*

$$\Sigma_\theta \cap \bigsqcup_k^p T_k^2,$$

which can represent the so-called "vanishing cycles" according to Novikov, does not exceed the constant $C_2(Vol(M), inj(M), H_0)||[\Sigma_\theta]||_{Th}$,

where $\Sigma_\theta = u^{-1}\theta$ and $T_k^2,\ k = 1, \ldots, p$, be tori bounded the Reeb components of $\mathcal{F}$.

Theorem

Let M be a closed oriented irreducible 3-Manifold with sectional curvature $K \leq K_0 \geq 0$ and $\mathcal{F}$ be a two-dimensional transversely oriented foliation $\mathcal{F}$ whose leaves have the modulus of mean curvature bounded above by the fixed positive constant H_0. The number of Reeb components of the foliation $\mathcal{F}$ does not exceed $\frac{4H_0 Vol(M)}{\sqrt{3}C_0^2}$, where

$$C_0 := \left\{ \begin{array}{ll} 2\min\{\min\{inj(M), \frac{\pi}{2\sqrt{K_0}}\}, \frac{1}{\sqrt{K_0}} \operatorname{arccot} \frac{H_0}{\sqrt{K_0}}\}, & \text{if } K_0 > 0 \\ 2\min\{inj(M), \frac{1}{H_0}\}, & \text{if } K_0 = 0 \end{array} \right\}.$$

In the case $K \equiv -1$ and $H_0 > 1$,
$C_0 = 2\min\{inj(M), (\coth)^{-1}(H_0)\}$.

The geometric meaning of the constant C_0 is that it is the systole $sys(\mathcal{F})$ of the foliation $\mathcal{F}$ if the set of essential integral loops (that is, loops belonging to some fiber $\mathcal{F}$ and not contractible inside it) is non-empty.

Thank you for your attention!