Nijenhuis Geometry

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Albert Nijenhuis

Albert Nijenhuis (November 21, 1926 – February 13, 2015),

Dutch-American mathematician who specialised in differential geometry and the theory of deformations in algebra and geometry, and later worked in combinatorics.

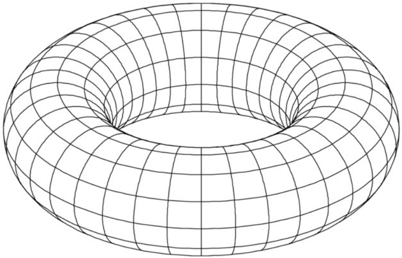
Alma mater: University of Amsterdam

Doctoral advisor: Prof. Jan Arnoldus Schouten

https://en.wikipedia.org/wiki/Albert_Nijenhuis

What is GEOMETRY?

Space, manifold M^n $+$ Structure



Structure is usually defined by means of a tensor, like g_{ij}, ω_{ij}, or P^{ij}

Naively, in coordinates, the geometric structure is defined by means of a matrix $A = \Big(a_{ij}(x)\Big)$ whose entries depend on coordinates $x = (x^1, \dots, x^n)$ and satisfy some algebraic and differential conditions.

Nijenhuis geometry. Our motivation

Definition

By Nijenhuis operators we understand (1, 1)-tensors $L = \big(L^i_j(x)\big)$ with vanishing Nijenhuis torsion:

$$\mathcal{N}_L(\xi, \eta) = L^2[\xi, \eta] + [L\xi, L\eta] - L[L\xi, \eta] - L[\xi, L\eta] = 0.$$

A manifold M endowed with such an operator it is called a Nijenhuis manifold.

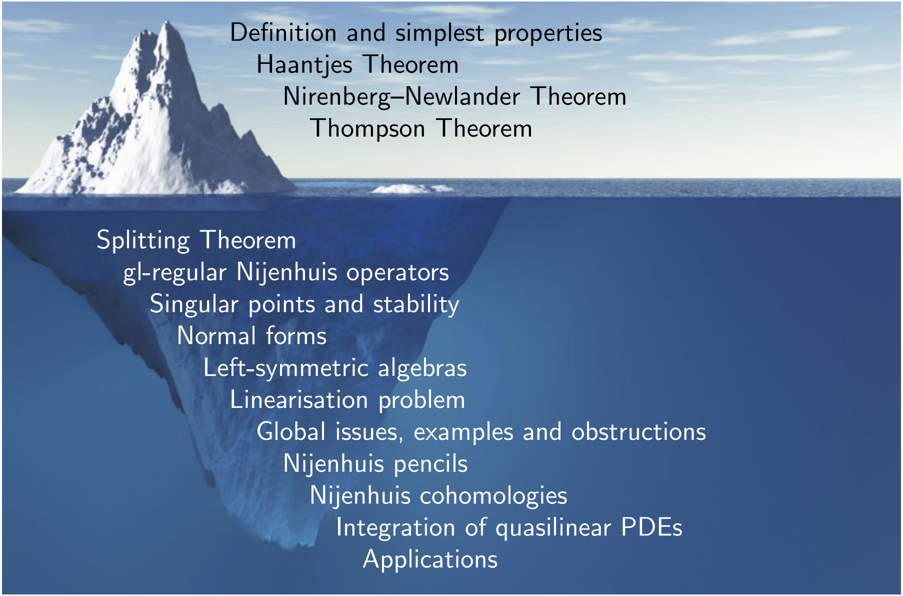
Motivation

- Riemannian, Kähler, symplectic, Poisson... Nijenhuis geometry is the next natural candidate to continue this list.
- In the context of the bi-Hamiltonian formalism, Nijenhuis operators occur as recursion operators.
- In the theory of integrable geodesic flows, projectively equivalent Riemannian metrics are related by means of a Nijenhuis operator.
- In topology of integrable systems, singularities of Lagrangian fibrations related to bi-Hamiltonian systems correspond to singular points of the corresponding Nijenhuis recursion operators.
- In integrable systems on Lie algebras, algebraic Nijenhuis operators are used in the study of Lie-Poisson pencils.

Elementary examples

- Constant operator:
$$L(x) = \left(L^i_j\right)$$
with L^i_j being constant for all i, j
- Scalar operator:
$$L(x) = f(x) \cdot \mathrm{Id},$$
where $f(x)$ is an arbitrary smooth function
- Complex structure
- $L(x) = \begin{pmatrix} x_1 & & & \\ & x_2 & & \\ & & \ddots & \\ & & & x_n \end{pmatrix}$
- $L(x) = \begin{pmatrix} x_5 & x_4 & x_3 & x_2 & x_1 \\ & x_5 & x_4 & x_3 & x_2 \\ & & x_5 & x_4 & x_3 \\ & & & x_5 & x_4 \\ & & & & x_5 \end{pmatrix}$

Nijenhuis Geometry



Research agenda

The ultimate goal of our research programme is to answer three fundamental questions:

(A) Local description: to what form can one bring a Nijenhuis operator near almost every point by a local coordinate change?

(B) Singular points: what does it mean for a point to be generic or singular in the context of Nijenhuis geometry? What singularities are non-degenerate/stable? How do Nijenhuis operators behave near non-degenerate and stable singular points?

(C) Global properties: what restrictions on a Nijenhuis operator are imposed by the topology of the underlying manifold? And conversely, what are topological obstructions to a Nijenhuis manifold carrying a Nijenhuis operator with specific properties?

as well as to work on

(D) Applications of Nijenhuis Geometry: in geometry, algebra and mathematical physics

Fundamental property of the characteristic polynomial of a Nijenhuis operator

Here and below L denotes a Nijenhuis operator.

Theorem

Let $\sigma_1, \dots, \sigma_n$ be the coefficients of the characteristic polynomial of L:

$$\chi(t) = \det(t \cdot \mathrm{Id} - L(x)) = t^n - \sum_{k=1}^{n} \sigma_k(x)\, t^{n-k}.$$

Then in any local coordinate system $x_1, \dots, x_n$ the following matrix relation hold:

$$J(x)\, L(x) = S_\chi(x)\, J(x), \quad \textit{where } S_\chi(x) = \begin{pmatrix} \sigma_1(x) & 1 & & \\ \vdots & 0 & \ddots & \\ \sigma_{n-1}(x) & \vdots & \ddots & 1 \\ \sigma_n(x) & 0 & \dots & 0 \end{pmatrix}$$

and $J(x) = \left(\frac{\partial \sigma_i}{\partial x_j}\right)$ is the Jacobi matrix of the collection of functions $\sigma_1, \dots, \sigma_n$ w.r.t. local coordinates $x_1, \dots, x_n$.

Two corollaries

Corollary

Assume that the coefficients of the characteristic polynomial of L are functionally independent almost everywhere on M. Then L can be uniquely reconstructed from them:

$$L(x) = J^{-1}(x) S_\chi(x) J(x).$$

Corollary

Assume that the coefficients of the characteristic polynomial of L are functionally independent in a neighbourhood of a point $\mathrm{p} \in M$ *(i.e.,* $\det J(x) \neq 0$*)* [1]. *Then there exists a local coordinate system $u^1, \dots, u^n$ in which L takes the following form*

$$L(u) = \begin{pmatrix} u^1 & 1 & & \\ \vdots & 0 & \ddots & \\ u^{n-1} & \vdots & \ddots & 1 \\ u^n & 0 & \dots & 0 \end{pmatrix}.$$

[1] Such points are called differentially non-degenerate.

Left-symmetric algebras and linearisation

Consider an operator $L = \left(a^i_{jk}x^k\right)$ whose components are linear functions in coordinates $x^1, \dots, x^n$.

Observation. L is Nijenhuis if and only if a^i_{jk} are structure constants of a left-symmetric algebra.

Reminder: An algebra $(\mathfrak{a}, *)$ is called *left-symmetric* (or *pre-Lie*) if:

$$\xi * (\eta * \zeta) - (\xi * \eta) * \zeta = \eta * (\xi * \zeta) - (\eta * \xi) * \zeta, \quad \text{for all } \xi, \eta, \zeta \in \mathfrak{a}.$$

Linearisation of a Nijenhuis operator at a singular point. Let L be a Nijenhuis operator in a neighbourhood of a point $q \in M$ such that $L(q) = 0$. In local coordinates $x = (x_1, \dots, x_n)$, centred at q, we can expand $L(x)$ into Taylor series:

$$L(x) = 0 + L_1(x) + L_2(x) + \dots$$

where L_k is homogeneous of degree k in $x_1, \dots, x_n$.
Then the first term $L_1 = L_{\text{lin}}$ defines a Nijenhuis operator in variables $x_1, \dots, x_n$, called the linearisation of L at q.
Conclusion:
Left symmetric algebras $\leftrightarrow$ linearisations of Nijenhuis structures

Linearisability and non-degenerate LSA and Lie algebras

Definition

A left-symmetric algebra $\mathfrak{a}$ is *non-degenerate* if every Nijenhuis operator L such that $L_{\text{lin}} \simeq \mathfrak{a}$, is linearisable, i.e., L is isomorphic to L_{lin}.

Theorem (A. Konyaev)

In dimension two, there are 12 types of real LSAs, six of which are non-degenerate in the smooth sense (with parameters appropriately chosen). In the real analytic case, the list of non-degenerate LSAs is different (slightly larger).

Theorem (Real analytic or formal)

Let $L(x) = L_{\mathrm{lin}}(x) + \dots$ *with*

$$L_{\text{lin}}(x) = \operatorname{diag}(x_1, x_2, \dots, x_n).$$

Then $L(x)$ *is linearisable, i.e., the diagonal LSA is non-degenerate. In other words, there exists a real analytic (formal) change of variables* $x \mapsto y$ *such that in the new coordinates*

$$L(y) = \operatorname{diag}(y_1, y_2, \dots, y_n).$$

Regular Nijenhuis manifolds

Definition

An operator L is called gl-*regular*, if each eigenvalue of L possesses only one eigenvector (up to proportionality). Equivalently, the operators $\mathrm{Id}, L, L^2, \dots, L^{n-1}$ are linearly independent.

Question: What is a local structure of a gl-regular Nijenhuis manifold?

Theorem (Real analytic case)

Let L be a gl-regular Nijenhuis operator. Then there exist local coordinate systems $u = (u_1, \dots, u_n)$ and $v = (v_1, \dots, v_n)$ in which L reduces to the first and second companion forms:

$$L(u) = \begin{pmatrix} \sigma_1 & 1 & & \\ \vdots & 0 & \ddots & \\ \sigma_{n-1} & \vdots & \ddots & 1 \\ \sigma_n & 0 & \dots & 0 \end{pmatrix} \quad \textit{and} \quad L(v) = \begin{pmatrix} 0 & 1 & & \\ \vdots & \ddots & \ddots & \\ 0 & \dots & 0 & 1 \\ \sigma_n & \sigma_{n-1} & \dots & \sigma_1 \end{pmatrix},$$

where σ_i are the coefficients of the characteristic polynomial of L in the corresponding coordinate system.

Some global results

Theorem

A Nijenhuis operator on a closed connected manifold cannot have non-constant complex eigenvalues.

Theorem

A Nijenhuis operator on a closed connected manifold cannot have differentially non-degenerate critical points.

Theorem

1. *Let* M^2 *be an orientable closed surface and* $\mathsf{M}^2 \not\simeq \mathsf{T}^2$ *(torus).* *Then on* M^2 *there are no* gl*-regular Nijenhuis operators except for* $L = \alpha \operatorname{Id} + \beta J$*, where* J *is a complex structure and* $\alpha, \beta \in \mathbb{R}$*,* $\beta \neq 0$*.*

2. *Let* M^2 *be a non-orientable closed surface and* $\mathsf{M}^2 \not\simeq \mathsf{K}^2$ *(Klein bottle).* *Then on* M^2 *there are no* gl*-regular Nijenhuis operators at all.*

Applications to geodesically equivalent metrics

Two (pseudo)-Riemannian metrics g and $\bar{g}$ are called *geodesically equivalent* if they share the same (unparametrised) geodesics.

Observation (Sinjukov). A manifold endowed with a pair of such metrics carries a natural Nijenhuis structure defined by the operator

$$L = \left| \frac{\det \bar{g}}{\det g} \right|^{\frac{1}{n+1}} \bar{g}^{-1} g.$$

The *geodesic equivalence condition* in terms of L:

$$\nabla_\xi L = \tfrac{1}{2}\big(u \otimes \mathrm{d}\,\mathrm{tr}\, L + (u \otimes \mathrm{d}\,\mathrm{tr}\, L)^*\big).$$

Here L serves as a partner of the metric g: we say that g and L are *geodesically compatible*, if this relation holds.

Important open problem. What happens *at singular points* (i.e., at those where the algebraic type of L changes, e.g., the eigenvalues of L collide)? Which singularities are allowed?

Theorem

Let L be a gl-regular real analytic Nijenhuis operator. Then (locally) there exists a pseudo-Riemannian metric g that is geodesically compatible with L. Moreover, such a metric g can be defined explicitly in terms of the second companion form of L.

Recall that an operator M is called a *symmetry* of L if these operators commute in algebraic sense, i.e. $ML = LM$, and the following relation holds:

$$M[L\xi, \xi] + L[\xi, M\xi] - [L\xi, M\xi] = 0 \quad \text{for any vector field } \xi.$$

A symmetry M is called *strong* if

$$\langle L, M\rangle(\xi, \eta) \stackrel{\text{def}}{=} M[L\xi, \eta] + L[\xi, M\eta] - [L\xi, M\eta] - LM[\xi, \eta] = 0 \quad \text{for all } \xi, \eta.$$

Theorem

Let L and g be geodesically compatible. Assume that M is g-self-adjoint and is a strong symmetry of L, then L and gM are geodesically compatible.

Moreover, if L is gl-regular, then every metric $\tilde{g}$ geodesically compatible with L is of the form $\tilde{g} = gM$, where M is a (strong) symmetry of L.

Open problems

- Describe the topology of closed $\mathfrak{gl}$-regular Nijenhuis manifolds.
- Construct real analytic examples of Nijenhuis operators on closed two-dimensional surfaces whose eigenvalues are real and generically distinct.
- Describe/classify maximal Nijenhuis pencils.
- Local classification of gl-regular Nijenhuis operators.
- Classify left-symmetric algebras of low dimension.
- Find/construct examples of non-degenerate LSAs of arbitrary dimension.
- Describe/classify LSAs with algebraically independent coefficients of the characteristic polynomial.
- etc.

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Thanks for your attention