Stability of minimal surfaces in the sub-Riemannian manifold $E(2)$

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Theorem (O.V. Pogorelov, M. do Carmo and K.K. Peng, D. Fischer-Colbrie and R. Schoen, [1])

A complete connected minimal surface in the three-dimensional Euclidean space is stable if and only if it is a plane.

This result generalizes the classical theorem of S.N. Bernstein which states that any complete minimal graph is a plane. The notion of a minimal surface in a sub-Riemannian manifold was introduced in [2]. Such surfaces and their stability were studied for various sub-Riemannian geometries (a short survey is in [4]).
A **sub-Riemannian manifold** is a smooth manifold $M$ together with a completely non-integrable smooth distribution $\mathcal{H}$ on $M$ (it is called a *horizontal distribution*) and a smooth field of Euclidean scalar products $\langle \cdot , \cdot \rangle_\mathcal{H}$ on $\mathcal{H}$ (it is called a *sub-Riemannian metric*). Example: three-dimensional Heisenberg group $\mathbb{H}^1$. This is the space $\mathbb{R}^3$ with coordinates $(x, y, z)$ and with the following basis of left-invariant vector fields defined by the Lie group structure:

$$X_1 = \frac{\partial}{\partial x} - y \frac{\partial}{\partial z}, \quad X_2 = \frac{\partial}{\partial y} + x \frac{\partial}{\partial z}, \quad X_3 = \frac{\partial}{\partial z}.$$ 

Let $\langle \cdot , \cdot \rangle$ be a Riemannian metric on $\mathbb{H}^1$ such that $\{X_1, X_2, X_3\}$ is an orthonormal frame. Then the horizontal distribution $\mathcal{H}$ is spanned by $\{X_1, X_2\}$ and the sub-Riemannian metric $\langle \cdot , \cdot \rangle_\mathcal{H}$ is the restriction of $\langle \cdot , \cdot \rangle$ to $\mathcal{H}$. 
Let $\Sigma$ be a smooth oriented surface in a three-dimensional sub-Riemannian manifold $M$, whose sub-Riemannian metric $\langle \cdot, \cdot \rangle_\mathcal{H}$ is a restriction of some Riemannian metrics on $M$ to $\mathcal{H}$. The singular set $\Sigma_0$ of this surface consists of points $p \in \Sigma$ such that the tangent plane $T_p \Sigma = \mathcal{H}_p$. If $N$ is the unit normal field $\Sigma$ in the Riemannian sense, then the singular set can be described as

$$\Sigma_0 = \{ p \in \Sigma \mid N_h(p) = 0 \},$$

where $N_h$ denotes the orthogonal projection of $N$ onto $\mathcal{H}$. The sub-Riemannian area of a domain $D \subset \Sigma$ is

$$A(D) = \int_D |N_h| \, d\Sigma,$$

where $d\Sigma$ is the Riemannian area form of $\Sigma$. 
The normal variation of the surface \( \Sigma \) defined by a smooth function \( u \) is the map

\[
\varphi: \Sigma \times I \to M: \varphi_s(p) = \exp_p(su(p)N(p)),
\]

where \( I \) is an open neighborhood of 0 in \( \mathbb{R} \) and \( \exp_p \) is the Riemannian exponential map in \( p \). In other words, we construct the variation in the traditional Riemannian way by drawing the geodesic through each point \( p \in \Sigma \) in the direction of the normal vector \( u(p)N(p) \). Denote

\[
A(s) = \int_{\Sigma_s} |N_h| d\Sigma_s,
\]

where \( \Sigma_s = \varphi_s(\Sigma) \). Then \( A'(0) \) is called the first (normal) area variation defined by \( \varphi \), and \( A''(0) \) is called the second one.
A surface $\Sigma$ is called \textit{minimal} if $A'(0) = 0$ for any normal variations with compact support in $\Sigma \setminus \Sigma_0$. 
Note that here we also follow the Riemannian tradition by defining minimal surfaces as stationary points of the sub-Riemannian area functional.
A minimal surface $\Sigma$ is called \textit{stable} if $A''(0) \geq 0$ for any normal variations with compact support in $\Sigma \setminus \Sigma_0$. 
The following Bernstein type result is known for $\mathbb{H}^1$:

\textbf{Theorem (A. Hurtado, M. Ritoré, C. Rosales, [3])}

A complete connected minimal surface with the empty singular set in the sub-Riemannian three-dimensional Heisenberg group is stable if and only if it is a vertical Euclidean plane.
The manifold $E(2)$ is the universal covering of the proper motions group of the Euclidean plane. This is the space $\mathbb{R}^3$ with coordinates $(x, y, z)$ (where $(x, y)$ and $z$ correspond to the translation vector and the rotation angle respectively) and with the following basis of left-invariant vector fields defined by the Lie group structure:

$$X_1 = \cos z \frac{\partial}{\partial x} + \sin z \frac{\partial}{\partial y}, \quad X_2 = \frac{\partial}{\partial z}, \quad X_3 = \sin z \frac{\partial}{\partial x} - \cos z \frac{\partial}{\partial y}.$$ 

Let $\langle \cdot, \cdot \rangle$ be a Riemannian metric on $E(2)$ such that $\{X_1, X_2, X_3\}$ is an orthonormal frame. Note that it is Euclidean. Then the horizontal distribution $\mathcal{H}$ is spanned by $\{X_1, X_2\}$ and the sub-Riemannian metric $\langle \cdot, \cdot \rangle_\mathcal{H}$ is the restriction of the Euclidean metric $\langle \cdot, \cdot \rangle$ to $\mathcal{H}$. 
Now let $\Sigma$ be a smooth oriented surface in $\widetilde{E}(2)$. Let us introduce some additional notation.

The *horizontal Gauss map* $\nu_h = \frac{N_h}{|N_h|}$ is defined on the regular part $\Sigma \setminus \Sigma_0$ of $\Sigma$. The *characteristic vector field* $Z$ is the right angle rotation of $\nu_h$ in $\mathcal{H}_p$ (in the orientation defined by $N(p)$) and is also defined on $\Sigma \setminus \Sigma_0$. Denote $S = \langle N, X_3 \rangle \nu_h - |N_h|X_3 \in T_p \Sigma$. The vector fields $Z$ and $S$ then form an orthonormal frame on $\Sigma \setminus \Sigma_0$.

Let $\nabla$ denote the Riemannian covariant derivative, and $B$ denote the Weingarten operator of $\Sigma$ with respect to $N$ that is defined by $B(W) = -\nabla_W N$ for any tangent vector field $W$ on $\Sigma$. 
Theorem (First and second variation formulae)

Let $\Sigma$ be a surface in $\widetilde{E}(2)$. Then its first normal area variation defined by a smooth function $u$ with compact support equals

$$A'(0) = \int_{\Sigma \setminus \Sigma_0} |N_h|^{-1} ( - \langle B(Z), Z \rangle + \langle N, X_3 \rangle \langle \nabla_{\nu_h} X_3, \nu_h \rangle ) u \, d\Sigma.$$

If $\Sigma$ is minimal, then its normal area variation defined by a smooth function $u$ with compact support equals

$$A''(0) = \int_{\Sigma \setminus \Sigma_0} -2|N_h| \langle B(Z), S \rangle^2 u^2 + 2|N_h| \langle B(Z), Z \rangle \langle B(S), S \rangle u^2 +$$

$$+2|N_h| \langle B(Z), Z \rangle u^2 (\langle B(S), S \rangle + \langle B(Z), Z \rangle) -$$

$$-2 \langle N, X_3 \rangle \langle B(S), Z \rangle Z(u) u + |N_h|^{-1} (Z(u) + \langle N, X_3 \rangle |N_h| \langle \nabla_{\nu_h} X_3, Z \rangle u)^2 -$$

$$-2 |N_h|^2 \langle \nabla_{\nu_h} X_3, \nu_h \rangle u - |N_h|^3 \langle \nabla_{\nu_h} X_3, \nu_h \rangle^2 u^2 \, d\Sigma.$$
Corollary (Minimality criterion)
A surface $\Sigma$ in $\widetilde{E}(2)$ is minimal if and only if

$$\langle B(Z), Z \rangle - \langle N, X_3 \rangle \langle \nabla_{\nu_h} X_3, \nu_h \rangle = 0.$$

Proposition
A Euclidean plane in $\widetilde{E}(2)$ is minimal if and only if it is a horizontal or vertical plane. All minimal Euclidean planes in $\widetilde{E}(2)$ are stable.
Corollary (Minimality criterion for graphs)

Let a surface in $\widetilde{E}(2)$ be defined by the equation $y = f(x, z)$. It is minimal if and only if

$$
- \cos^2 z f^2_z f_{xx} + (2 \cos^2 z f_x f_z - \sin^2 z f_z)f_{xz} +

+ (-\cos^2 z f_x^2 + \sin^2 z f_x - \sin^2 z)f_{zz} -

- \cos z \sin z f_x^2 f_z + (-\cos^2 z + \sin^2 z)f_x f_z + \sin z \cos z f_z = 0.
$$

Some non-planar solutions are $y = A \cos z + B$ and $y = x + A(\sin z + \cos z) + B$, where $A, B \in \mathbb{R}$. Similar equations can be written for $x = f(y, z)$ and $z = f(x, y)$. Thus, the minimality of a surface in the Riemannian sense does not imply its sub-Riemannian minimality, and vice versa.
We will call a surface $\Sigma$ in a three-dimensional sub-Riemannian manifold vertical if $T_p\Sigma \perp \mathcal{H}_p$ for each $p \in \Sigma$. In particular, such surfaces do not contain singular points.

**Theorem**

Any complete connected vertical minimal surface in $E(2)$ is either a horizontal Euclidean plane $z = C$ or a standard helicoid $x \cos z + y \sin z = 0$ translated parallelly along the $(x, y)$-plane. Helicoids are unstable.

From this we obtain the following partial Bernstein type result:

**Corollary**

A complete connected vertical minimal surface in $E(2)$ is stable if and only if it is a horizontal Euclidean plane.
Proof. For any vertical surface $\langle N, X_3 \rangle = 0$ and $|N_h| = 1$, thus $S = -X_3$, so this surface consists of integral trajectories of $X_3$. They are Euclidean straight lines with direction vectors $(\sin z, \cos z, 0)$. Therefore, the surface is ruled and can be parameterized locally as

$$r(\rho, \varphi) = (x(\varphi), y(\varphi), z(\varphi)) + \rho(\sin z(\varphi), -\cos z(\varphi), 0),$$

where $(x, y, z)$ is a naturally parameterized integral trajectory of the field $S$, so the condition $x' \sin z - y' \cos z = 0$ holds. For vertical surfaces the minimality criterion has the form $\langle B(Z), Z \rangle = 0$, thus for our surface

$$(z''x' - z'x'') \cos z + (z''y' - z'y'') \sin z = 0.$$ 

We get the required classes of surfaces by solving this equation.
For vertical surfaces the second variation formula can be rewritten as

\[ A''(0) = \int_{\Sigma} Z(u)^2 + (-2 \langle B(Z), S \rangle^2 + \langle B(Z), S \rangle (1-2 \langle \nu_h, X_1 \rangle^2))u^2 \ d\Sigma. \]

For the parameterization

\[ r(\rho, \varphi) = (\rho \sin \varphi, -\rho \cos \varphi, \varphi) \]

of the helicoid we have

\[ A''(0) = \int_{\Sigma} \frac{1}{1 + \rho^2} (u^2_\varphi - u^2) \sqrt{1 + \rho^2} \ d\rho d\varphi. \]

Then \( A''(0) < 0 \) for \( u = \cos \rho (\cos \varphi + \sqrt{2}) \), \( \rho \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \), \( \varphi \in \left[-\frac{3\pi}{4}, \frac{3\pi}{4}\right] \).
Thank you!
Дякую за увагу!

