Stability of minimal surfaces in the sub-Riemannian manifold $\widetilde{E(2)}$

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Theorem (O.V. Pogorelov, M. do Carmo and K.K. Peng, D. Fischer-Colbrie and R. Schoen, [1])

A complete connected minimal surface in the three-dimensional Euclidean space is stable if and only if it is a plane.

This result generalizes the classical theorem of S.N. Bernstein which states that any complete minimal graph is a plane. The notion of a minimal surface in a sub-Riemannian manifold was introduced in [2]. Such surfaces and their stability were studied for various sub-Riemannian geometries (a short survey is in [4]).

A *sub-Riemannian manifold* is a smooth manifold M together with a completely non-integrable smooth distribution $\mathcal{H}$ on M (it is called a *horizontal distribution*) and a smooth field of Euclidean scalar products $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ on $\mathcal{H}$ (it is called a *sub-Riemannian metric*). Example: three-dimensional Heisenberg group $\mathbb{H}^1$. This is the space $\mathbb{R}^3$ with coordinates (x, y, z) and with the following basis of left-invariant vector fields defined by the Lie group structure:

$$X_1 = \frac{\partial}{\partial x} - y\frac{\partial}{\partial z}, X_2 = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}, X_3 = \frac{\partial}{\partial z}.$$

Let $\langle \cdot, \cdot \rangle$ be a Riemannian metric on $\mathbb{H}^1$ such that $\{X_1, X_2, X_3\}$ is an orthonormal frame. Then the horizontal distribution $\mathcal{H}$ is spanned by $\{X_1, X_2\}$ and the sub-Riemannian metric $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ is the restriction of $\langle \cdot, \cdot \rangle$ to $\mathcal{H}$.

Let Σ be a smooth oriented surface in a three-dimensional sub-Riemannian manifold M, whose sub-Riemannian metric $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ is a restriction of some Riemannian metrics on M to $\mathcal{H}$. The *singular set* Σ_0 of this surface consists of points $p \in \Sigma$ such that the tangent plane $T_p\Sigma = \mathcal{H}_p$. If N is the unit normal field Σ in the Riemannian sense, then the singular set can be described as

$$\Sigma_0 = \{p \in \Sigma \mid N_h(p) = 0\},$$

where N_h denotes the orthogonal projection of N onto $\mathcal{H}$.
The *sub-Riemannian area* of a domain $D \subset \Sigma$ is

$$A(D) = \int_D |N_h| \, d\Sigma,$$

where $d\Sigma$ is the Riemannian area form of Σ.

The *normal variation* of the surface Σ defined by a smooth function u is the map

$$\varphi\colon \Sigma \times I \to M\colon \varphi_s(p) = \exp_p(su(p)N(p)),$$

where I is an open neighborhood of 0 in $\mathbb{R}$ and $\exp_p$ is the Riemannian exponential map in p. In other words, we construct the variation in the traditional Riemannian way by drawing the geodesic through each point $p \in \Sigma$ in the direction of the normal vector $u(p)N(p)$. Denote

$$A(s) = \int_{\Sigma_s} |N_h| \, d\Sigma_s,$$

where $\Sigma_s = \varphi_s(\Sigma)$. Then $A'(0)$ is called the *first (normal) area variation* defined by φ, and $A''(0)$ is called the *second* one.

A surface Σ is called *minimal* if $A'(0) = 0$ for any normal variations with compact support in $\Sigma \setminus \Sigma_0$.
Note that here we also follow the Riemannian tradition by defining minimal surfaces as stationary points of the sub-Riemannian area functional.
A minimal surface Σ is called *stable* if $A''(0) \geq 0$ for any normal variations with compact support in $\Sigma \setminus \Sigma_0$.
The following Bernstein type result is known for $\mathbb{H}^1$:

Theorem (A. Hurtado, M. Ritoré, C. Rosales, [3])

A complete connected minimal surface with the empty singular set in the sub-Riemannian three-dimensional Heisenberg group is stable if and only if it is a vertical Euclidean plane.

The manifold $\widetilde{E(2)}$ is the universal covering of the proper motions group of the Euclidean plane. This is the space $\mathbb{R}^3$ with coordinates (x, y, z) (where (x, y) and z correspond to the translation vector and the rotation angle respectively) and with the following basis of left-invariant vector fields defined by the Lie group structure:

$$X_1 = \cos z \frac{\partial}{\partial x} + \sin z \frac{\partial}{\partial y}, X_2 = \frac{\partial}{\partial z}, X_3 = \sin z \frac{\partial}{\partial x} - \cos z \frac{\partial}{\partial y}.$$

Let $\langle \cdot, \cdot \rangle$ be a Riemannian metric on $\widetilde{E(2)}$ such that $\{X_1, X_2, X_3\}$ is an orthonormal frame. Note that it is Euclidean. Then the horizontal distribution $\mathcal{H}$ is spanned by $\{X_1, X_2\}$ and the sub-Riemannian metric $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ is the restriction of the Euclidean metric $\langle \cdot, \cdot \rangle$ to $\mathcal{H}$.

Now let Σ be a smooth oriented surface in $\widetilde{E(2)}$. Let us introduce some additional notation.
The *horizontal Gauss map* $\nu_h = \frac{N_h}{|N_h|}$ is defined on the regular part $\Sigma \setminus \Sigma_0$ of Σ. The *characteristic vector field* Z is the right angle rotation of ν_h in $\mathcal{H}_p$ (in the orientation defined by $N(p)$) and is also defined on $\Sigma \setminus \Sigma_0$. Denote $S = \langle N, X_3 \rangle \, \nu_h - |N_h| X_3 \in T_p\Sigma$. The vector fields Z and S then form an orthonormal frame on $\Sigma \setminus \Sigma_0$.
Let ∇ denote the Riemannian covariant derivative, and B denote the Weingarten operator of Σ with respect to N that is defined by $B(W) = -\nabla_W N$ for any tangent vector field W on Σ.

Theorem (First and second variation formulae)

Let Σ be a surface in $\widetilde{E(2)}$. Then its first normal area variation defined by a smooth function u with compact support equals

$$A'(0) = \int_{\Sigma \setminus \Sigma_0} |N_h|^{-1} \left(-\langle B(Z), Z\rangle + \langle N, X_3\rangle \langle \nabla_{\nu_h} X_3, \nu_h\rangle\right) u \, d\Sigma.$$

If Σ is minimal, then its normal area variation defined by a smooth function u with compact support equals

$$A''(0) = \int_{\Sigma \setminus \Sigma_0} -2|N_h| \left\langle B(Z), S\right\rangle^2 u^2 + 2|N_h| \left\langle B(Z), Z\right\rangle \left\langle B(S), S\right\rangle u^2 +$$

$$+2|N_h| \left\langle B(Z), Z\right\rangle u^2(\langle B(S), S\rangle + \langle B(Z), Z\rangle) -$$

$$-2 \left\langle N, X_3\right\rangle \left\langle B(S), Z\right\rangle Z(u)u + |N_h|^{-1}(Z(u) + \langle N, X_3\rangle |N_h| \left\langle \nabla_{\nu_h} X_3, Z\right\rangle u)^2 -$$

$$-2|N_h|^2 \left\langle \nabla_{\nu_h} X_3, \nu_h\right\rangle u - |N_h|^3 \left\langle \nabla_{\nu_h} X_3, \nu_h\right\rangle^2 u^2 \, d\Sigma.$$

Corollary (Minimality criterion)

A surface Σ in $\widetilde{E(2)}$ is minimal if and only if

$$\langle B(Z), Z\rangle - \langle N, X_3\rangle \langle \nabla_{\nu_h} X_3, \nu_h\rangle = 0.$$

Proposition

A Euclidean plane in $\widetilde{E(2)}$ is minimal if and only if it is a horizontal or vertical plane. All minimal Euclidean planes in $\widetilde{E(2)}$ are stable.

Corollary (Minimality criterion for graphs)

Let a surface in $\widetilde{E(2)}$ be defined by the equation $y = f(x,z)$. It is minimal if and only if

$$-\cos^2 z\, f_z^2 f_{xx} + (2\cos^2 z\, f_x f_z - \sin^2 z f_z) f_{xz} +$$
$$+(-\cos^2 z\, f_x^2 + \sin^2 z\, f_x - \sin^2 z) f_{zz} -$$
$$-\cos z \sin z\, f_x^2 f_z + (-\cos^2 z + \sin^2 z) f_x f_z + \sin z \cos z\, f_z = 0.$$

Some non-planar solutions are $y = A\cos z + B$ and $y = x + A(\sin z + \cos z) + B$, where $A, B \in \mathbb{R}$. Similar equations can be written for $x = f(y,z)$ and $z = f(x,y)$.
Thus, the minimality of a surface in the Riemannian sense does not imply its sub-Riemannian minimality, and vice versa.

We will call a surface Σ in a three-dimensional sub-Riemannian manifold *vertical* if $T_p\Sigma \perp \mathcal{H}_p$ for each $p \in \Sigma$. In particular, such surfaces do not contain singular points.

Theorem

Any complete connected vertical minimal surface in $\widetilde{E(2)}$ is either a horizontal Euclidean plane $z = C$ or a standard helicoid $x\ cosz + y \sin z = 0$ translated parallelly along the (x, y)-plane. Helicoids are unstable.

From this we obtain the following partial Bernstein type result:

Corollary

A complete connected vertical minimal surface in $\widetilde{E(2)}$ is stable if and only if it is a horizontal Euclidean plane.

Proof. For any vertical surface $\langle N, X_3 \rangle = 0$ and $|N_h| = 1$, thus $S = -X_3$, so this surface consists of integral trajectories of X_3. They are Euclidean straight lines with direction vectors $(\sin z, \cos z, 0)$. Therefore, the surface is ruled and can be parameterized locally as

$$r(\rho, \varphi) = (x(\varphi), y(\varphi), z(\varphi)) + \rho(\sin z(\varphi), -\cos z(\varphi), 0),$$

where (x, y, z) is a naturally parameterized integral trajectory of the field S, so the condition $x' \sin z - y' \cos z = 0$ holds. For vertical surfaces the minimality criterion has the form $\langle B(Z), Z \rangle = 0$, thus for our surface

$$(z''x' - z'x'') \cos z + (z''y' - z'y'') \sin z = 0.$$

We get the required classes of surfaces by solving this equation.

For vertical surfaces the second variation formula can be rewritten as

$$A''(0) = \int_\Sigma Z(u)^2 + (-2\langle B(Z), S\rangle^2 + \langle B(Z), S\rangle (1-2\langle \nu_h, X_1\rangle^2))u^2 \, d\Sigma.$$

For the parameterization

$$r(\rho, \varphi) = (\rho \sin\varphi, -\rho\cos\varphi, \varphi)$$

of the helicoid we have

$$A''(0) = \int_\Sigma \frac{1}{1+\rho^2}(u_\varphi^2 - u^2)\sqrt{1+\rho^2}\, d\rho d\varphi.$$

Then $A''(0) < 0$ for $u = \cos\rho(\cos\varphi + \frac{\sqrt{2}}{2})$, $\rho \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, $\varphi \in [-\frac{3\pi}{4}, \frac{3\pi}{4}]$.

Thank you!

Дякую за увагу!

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