ON DIRECT LIMITS OF MINKOWSKI’S BALLS, DOMAINS, AND THEIR CRITICAL LATTICES

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1. Minkowski’s balls

We construct direct systems of Minkowski, Davis and Chebyshev-Cohn balls and domains, direct systems of their critical lattices and calculate their direct limits. By general (Minkowski’s) balls we mean (two-dimensional) balls in $\mathbb{R}^2$ of the form

$$D_p : |x|^p + |y|^p, p \geq 1.$$  (1)

The boundaries of the balls form a real manifold with boundary

$$|x|^p + |y|^p = 1, 1 \leq p \leq \infty.$$  (2)

Denote by $V(D_p)$ the volume (area) of $D_p.$

Proposition 2 ([1]). The volume of Minkowski ball $D_p$ is equal $4 \left( \frac{\Gamma(1+\frac{1}{p})}{\Gamma(1+\frac{2}{p})} \right)^2.$
3. Theorem on critical determinants of Minkowski’s balls and their critical lattices

From the proof of Minkowski’s conjecture [1, 2, 3, 4, 5, 8] in notations [8, 9] we have next expressions for critical determinants and their lattices:

**Theorem 4.** (1) \( \Delta(D_p) = \Delta^{(0)}_p = \Delta(p, \sigma_p) = \frac{1}{2} \sigma_p, \ 2 \leq p \leq p_0; \)

(2) \( \sigma_p = \left(2^p - 1\right)^{1/p}, \)

(3) \( \Delta(D_p) = \Delta^{(1)}_p = \Delta(p, 1) = 4^{1/p} \left(1 + \frac{1}{2^{1/p}}\right), \ 1 \leq p \leq 2, \ p \geq p_0, \)

(4) \( 2(1 - \tau_p)^p = 1 + \tau_p^p, \ 0 \leq \tau_p < 1, \)

here \( p_0 \) is a real number that is defined unique by conditions \( \Delta(p_0, \sigma_p) = \Delta(p_0, 1), \ 2.57 < p_0 < 2.58, \ p_0 \approx 2.5725 \)

For their critical lattices respectively \( \Lambda^{(0)}_p, \Lambda^{(1)}_p \) next conditions satisfy: \( \Lambda^{(0)}_p \) and \( \Lambda^{(1)}_p \) are two \( D_p \)-admissible lattices each of which contains three pairs of points on the boundary of \( D_p \) with the property that \((1, 0) \in \Lambda^{(0)}_p, \ (-2^{-1/p}, 2^{-1/p}) \in \Lambda^{(1)}_p, \)
5. Examples

**Example 6.** Lattices $\Lambda_p^{(0)}$ are two-dimensional lattices in $\mathbb{R}^2$ spanned by the vectors

\[
\begin{align*}
\lambda^{(1)} &= (1, 0), \\
\lambda^{(2)} &= \left(\frac{1}{2}, \frac{1}{2} \sigma_p\right).
\end{align*}
\]

The lattice $\Lambda_2^{(1)}$ is a two-dimensional lattice in $\mathbb{R}^2$ spanned by the vectors

\[
\begin{align*}
\lambda^{(1)} &= (-2^{-1/2}, 2^{-1/2}), \\
\lambda^{(2)} &= \left(\frac{\sqrt{6}-\sqrt{2}}{4}, \frac{\sqrt{6}+\sqrt{2}}{4}\right).
\end{align*}
\]
7. Balls and Domains

We consider balls of the form

\[ D_p : |x|^p + |y|^p \leq 1, \quad p \geq 1, \]

and call such balls with \( 1 < p < 2 \) Minkowski balls. Continuing this, we consider the following classes of balls and circles.

- **Davis balls**: \( |x|^p + |y|^p \leq 1 \) for \( p_0 > p \geq 2 \);
- **Chebyshev-Cohn balls**: \( |x|^p + |y|^p \leq 1 \) for \( p \geq p_0 \).

Let \( D \) be a fixed bounded symmetric about origin convex body (centrally symmetric convex body for short) with volume \( V(D) \).

**Proposition 8.** [6] If \( D \) is symmetric about the origin and convex, then \( 2D \) is convex and symmetric about the origin.

**Corollary 9.** Let \( m \) be integer \( m \geq 0 \) and \( n \) be natural greater \( m \). If \( 2^m D \) centrally symmetric convex body then \( 2^n D \) is again centrally symmetric convex body.

**Proof.** Induction.

We consider the following classes of balls (see above) and domains.

- **Minkowski domains**: \( 2^m D_p \), integer \( m \geq 1 \), for \( 1 \leq p < 2 \);
- **Davis domains**: \( 2^m D_p \), integer \( m \geq 1 \), for \( p_0 > p \geq 2 \);
- **Chebyshev-Cohn domains**: \( 2^m D_p \), integer \( m \geq 1 \), for \( p \geq p_0 \).

**Proposition 10.** Let \( m \) be integer, \( m \geq 1 \). If \( \Lambda \) is the critical lattice of the ball \( D_p \) than the sublattice \( \Lambda_{2^m} \) of index \( 2^m \) is the critical lattice of the domain \( 2^{m-1} D_p \).
11. **Direct systems**

Direct systems and direct limits were defined by Pontrygin [7]. The direct system of Minkowski balls and domains has the form (3), where the multiplication by 2 is the continuous mapping

\[ D_p \xrightarrow{2} 2D_p \xrightarrow{2} 2^2D_p \xrightarrow{2} \cdots \xrightarrow{2} 2^mD_p \xrightarrow{2} \cdots \]  

(3)

The direct system of critical lattices has the form (4), where the multiplication by 2 is the homomorphism of abelian groups

\[ \Lambda_p \xrightarrow{2} 2\Lambda_p \xrightarrow{2} 2^2\Lambda_p \xrightarrow{2} \cdots \xrightarrow{2} 2^m\Lambda_p \xrightarrow{2} \cdots \]  

(4)

In our considerations we have direct systems of Minkowski balls, Minkowski domains and direct systems of critical lattices with respective maps and homomorphisms. Let \( \mathbb{Q}_2 \) and \( \mathbb{Z}_2 \) be respectively the field of 2-adic numbers and its ring of integers. Denote the corresponding direct limits by \( D_p^{dirlim} \) and by \( \Lambda_p^{dirlim} \).
12. Direct limits

**Proposition 13.** $D_{p}^{\text{dirlim}} = \lim_{\rightarrow} 2^{m} D_{p} \in (\mathbb{Q}/\mathbb{Z}) D_{p} = (\bigcup_{m} \frac{1}{2^{m}} \mathbb{Z}/\mathbb{Z}) D_{p}$.

**Proposition 14.** $\Lambda_{p}^{\text{dirlim}} = \lim_{\rightarrow} 2^{m} \Lambda_{p} \in (\mathbb{Q}/\mathbb{Z}) \Lambda_{p} = (\bigcup_{m} \frac{1}{2^{m}} \mathbb{Z}/\mathbb{Z}) \Lambda_{p}$.
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References

Thank you!