On direct limits of Minkowski's balls, domains, and their critical lattices

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1. Minkowski's balls

We construct direct systems of Minkowski, Davis and Chebyshev-Cohn balls and domains, direct systems of their critical lattices and calculate their direct limits. By general (Minkowski's) balls we mean (two-dimensional) balls in $\mathbb{R}^2$ of the form

$$D_p : \ |x|^p + |y|^p . \ p \geq 1. \tag{1}$$

The boundaries of the balls form a real manifold with boundary

$$|x|^p + |y|^p = 1, \ 1 \leq p \leq \infty. \tag{2}$$

Denote by $V(D_p)$ the volume (area) of D_p.

Proposition 2 ([1]). *The volume of Minkowski ball D_p is equal* $4\frac{(\Gamma(1+\frac{1}{p}))^2}{\Gamma(1+\frac{2}{p})}$.

3. Theorem on critical determinants of Minkowski's balls and their critical lattices

From the proof of Minkowski's conjecture [1, 2, 3, 4, 5, 8] in notations [8, 9] we have next expressions for critical determinants and their lattices:

Theorem 4. (1) $\Delta(D_p) = \Delta_p^{(0)} = \Delta(p, \sigma_p) = \frac{1}{2}\sigma_p,\ 2 \le p \le p_0;$

(2) $\sigma_p = (2^p - 1)^{1/p},$

(3) $\Delta(D_p) = \Delta_p^{(1)} = \Delta(p, 1) = 4^{-\frac{1}{p}} \frac{1+\tau_p}{1-\tau_p},\ 1 \le p \le 2,\ p \ge p_0,$

(4) $2(1 - \tau_p)^p = 1 + \tau_p^p,\ 0 \le \tau_p < 1,$

here p_0 *is a real number that is defined unique by conditions* $\Delta(p_0, \sigma_p) = \Delta(p_0, 1),\ 2,57 < p_0 < 2,58,\ p_0 \approx 2.5725$

For their critical lattices respectively $\Lambda_p^{(0)},\ \Lambda_p^{(1)}$ *next conditions satisfy:* $\Lambda_p^{(0)}$ *and* $\Lambda_p^{(1)}$ *are two* D_p*-admissible lattices each of which contains three pairs of points on the boundary of* D_p *with the property that* $(1, 0) \in \Lambda_p^{(0)},\ (-2^{-1/p}, 2^{-1/p}) \in \Lambda_p^{(1)},$

5. Examples

Example 6. Lattices $\Lambda_p^{(0)}$ are two-dimensional lattices in $\mathbb{R}^2$ spanned by the vectors

$\lambda^{(1)} = (1, 0),$
$\lambda^{(2)} = (\frac{1}{2}, \frac{1}{2}\sigma_p).$

The lattice $\Lambda_2^{(1)}$ is a two-dimensional lattice in $\mathbb{R}^2$ spanned by the vectors

$\lambda^{(1)} = (-2^{-1/2}, 2^{-1/2}),$
$\lambda^{(2)} = (\frac{\sqrt{6}-\sqrt{2}}{4}, \frac{\sqrt{6}+\sqrt{2}}{4}).$

7. Balls and Domains

We consider balls of the form

$$D_p: \ |x|^p + |y|^p \leq 1, \ p \geq 1,$$

and call such balls with $1 < p < 2$ *Minkowski balls*. Continuing this, we consider the following classes of balls and circles.

- *Davis balls*: $|x|^p + |y|^p \leq 1$ for $p_0 > p \geq 2$;
- *Chebyshev-Cohn balls*: $|x|^p + |y|^p \leq 1$ for $p \geq p_0$;

Let D be a fixed bounded symmetric about origin convex body (*centrally symmetric convex body* for short) with volume $V(D)$.

Proposition 8. [6]. *If D is symmetric about the origin and convex, then $2D$ is convex and symmetric about the origin.*

Corollary 9. *Let m be integer $m \geq 0$ and n be natural greater m. If $2^m D$ centrally symmetric convex body then $2^n D$ is again centrally symmetrc convex body.*

Proof. Induction.
We consider the following classes of balls (see above) and domains.

- *Minkowski domains*: $2^m D_p$, integer $m \geq 1$, for $1 \leq p < 2$;
- *Davis domains*: $2^m D_p$, integer $m \geq 1$, for $p_0 > p \geq 2$;
- *Chebyshev-Cohn domains*: $2^m D_p$, integer $m \geq 1$, for $p \geq p_0$;

Proposition 10. *Let m be integer, $m \geq 1$. If Λ is the critical lattice of the ball D_p than the sublattice Λ_{2^m} of index 2^m is the critical lattice of the domain $2^{m-1} D_p$.*

11. Direct systems

Direct systems and direct limits were defined by Pontrygin [7]. The direct system of Minkowski balls and domains has the form (3), where the multiplication by 2 is the continuous mapping

$$D_p \xrightarrow{2} 2D_p \xrightarrow{2} 2^2D_p \xrightarrow{2} \cdots \xrightarrow{2} 2^mD_p \xrightarrow{2} \cdots \tag{3}$$

The direct system of critical lattices has the form (4), where the multiplication by 2 is the homomorphism of abelian groups

$$\Lambda_p \xrightarrow{2} 2\Lambda_p \xrightarrow{2} 2^2\Lambda_p \xrightarrow{2} \cdots \xrightarrow{2} 2^m\Lambda_p \xrightarrow{2} \cdots \tag{4}$$

In our considerations we have direct systems of Minkowski balls, Minkowski domains and direct systems of critical lattices with respective maps and homomorphisms. Let $\mathbb{Q}_2$ and $\mathbb{Z}_2$ be respectively the field of 2-adic numbers and its ring of integers. Denote the corresponding direct limits by D_p^{dirlim} and by Λ_p^{dirlim}.

12. Direct limits

Proposition 13. $D_p^{dirlim} = \varinjlim 2^m D_p \in (\mathbb{Q}_2/\mathbb{Z}_2)D_p = (\bigcup_m \frac{1}{2^m}\mathbb{Z}_2/\mathbb{Z}_2)D_p$.

Proposition 14. $\Lambda_p^{dirlim} = \varinjlim 2^m \Lambda_p \in (\mathbb{Q}_2/\mathbb{Z}_2)\Lambda_p = (\bigcup_m \frac{1}{2^m}\mathbb{Z}_2/\mathbb{Z}_2)\Lambda_p$.

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Thank you!