Semi-Fredholm theory in C^*-algebras

AGMA - conference

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Recall that if H is a Hilbert space and $F \in B(H)$ such that ImF is closed, then by the Banach open mapping theorem we have a decomposition

$$H = (\ker F)^{\perp} \oplus \ker F \xrightarrow{F} ImF \oplus (ImF)^{\perp} = H$$

with respect to which F has the matrix $\begin{bmatrix} F_1 & 0 \\ 0 & 0 \end{bmatrix}$, where F_1 is an isomorphism.

A bounded linear operator F operator on H is a semi-Fredholm if ImF is closed and either

$$\dim \ker F < \infty \text{ or } \dim(ImF)^{\perp} < \infty.$$

If $\dim \ker F < \infty$, then F is called an upper semi-Fredholm operator on H, whereas if $\dim(ImF)^{\perp} < \infty$, then F is called a lower semi-Fredholm operator on H. If F is both an upper and a lower semi-Fredholm operator on H, then F is said to be a Fredholm operator on H.

Now, Hilbert C^*-modules are a natural generalization of Hilbert spaces when the field of scalars is replaced by an arbitrary C^*-algebra. Fredholm theory on Hilbert C^*-modules as a generalization of the classical Fredholm theory on Hilbert spaces was started by Mishchenko and Fomenko. In [**MF**] they introduced the notion of a Fredholm operator on the standard Hilbert C^*-module and proved a generalization in this setting of some of the main results from the classical Fredholm theory. In [**IS1**], [**IS2**], [**IS3**], [**IS4**], [**IS5**] we went further in this direction and defined semi-Fredholm and semi-Weyl operators on Hilbert C^*-modules. We investigated and proved several properties of these new semi-Fredholm operators on Hilbert C^*-modules as a generalization of the results from the classical semi-Fredholm theory on Hilbert and Banach spaces.

If M is a Hilbert C^*-module and M_1, M_2 are two closed submodules of M, we write $M = M_1 \tilde{\oplus} M_2$ if $M_1 \cap M_2 = \{0\}$ and $M_1 + M_2 = M$.

The set of all adjointable, bounded, C^*-linear operators from M into M will be denoted by $B^a(M)$. It can be shown that $B^a(M)$ is a C^*-algebra.

The standard Hilbert module over a C^* -algebra $\mathcal{A}$ is $l^2(\mathcal{A})$ which we will denote by $H_{\mathcal{A}}$.

Definition

[**IS1**] [**MF**] Let $F \in B^a(H_{\mathcal{A}})$. We say that F is an upper semi-$\mathcal{A}$-Fredholm operator if there exists a decomposition

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}}$$

with respect to which F has the matrix

$$\left[\begin{array}{cc} F_1 & 0 \\ 0 & F_4 \end{array} \right],$$

where F_1 is an isomorphism, M_1, M_2, N_1, N_2 are closed submodules of $H_{\mathcal{A}}$ and N_1 is finitely generated. Similarly, we say that F is a lower semi-$\mathcal{A}$-Fredholm operator if all the above conditions hold except that in this case we assume that N_2 (and not N_1) is finitely generated.
If both N_1 and N_2 are finitely generated, we say that F is $\mathcal{A}$-Fredholm.

We set

$$\mathcal{M}\Phi_+(H_\mathcal{A}) = \{F \in B^a(H_\mathcal{A}) \mid F \text{ is upper semi-}\mathcal{A}\text{-Fredholm }\},$$

$$\mathcal{M}\Phi_-(H_\mathcal{A}) = \{F \in B^a(H_\mathcal{A}) \mid F \text{ is lower semi-}\mathcal{A}\text{-Fredholm }\},$$

$$\mathcal{M}\Phi(H_\mathcal{A}) = \{F \in B^a(H_\mathcal{A}) \mid F \text{ is } \mathcal{A}\text{-Fredholm operator on } H_\mathcal{A}\}.$$

Breuer started the development of Fredholm theory in von-Neumann algebras as a generalization of the classical Fredholm theory for operators on Hilbert spaces. In [**BR**] and [**BR2**] he introduced the notion of a Fredholm operator in a von Neumann algebra and established its main properties as a generalization in this setting of some well-known properties of the classical Fredholm operator on a Hilbert space.
Let us recall first the notion of Murray-Von Neumann equivalence.

Definition

[**KL**] Let $\mathcal{A}$ be an unital C^*-algebra.
In the set Proj($\mathcal{A}$) we define the equivalence relation:

$$p \sim q \Leftrightarrow \exists v \in \mathcal{A} \; vv^* = p, \; v^*v = q,$$

i.e. Murray - von Neumann equivalence.

Definition

[**BM**] Let $\mathcal{A}$ be a von Neumann algebra, let $Proj(\mathcal{A})$ be the set of all projections belonging to $\mathcal{A}$, and let $Proj_0(\mathcal{A})$ be the set of all finite projections in $\mathcal{A}$ (i.e. those projections that are not Murray von Neumann equivalent to any its proper subprojection).
The operator $T \in \mathcal{A}$ is said to be $\mathcal{A}$-Fredholm if the following holds.
(i) $P_{\ker T} \in Proj_0(\mathcal{A})$, where $P_{\ker T}$ is the projection to the subspace $\ker T$.
(ii) There is a projection $E \in Proj_0(\mathcal{A})$ such that $Im(I - E) \subseteq ImT$.
The second condition ensures that $P_{(ImT)^{\perp}}$ also belongs to $Proj_0(\mathcal{A})$.

Kečkić and Lazović in [**KL**] introduced an axiomatic approach to Fredholm theory by introducing the notion of a Fredholm type element in a unital C*-algebra. This notion is a generalization of C*-Fredholm operator on the standard Hilbert C*-module introduced by Mishchenko and Fomenko and of Fredholm operator on a properly infinite von Neumann algebra introduced by Breuer. They obtained then that the set of Fredholm type elements in a unital C*-algebra is open in the norm topology and invariant under perturbation by finite type elements. Also, they proved multiplicativity of the index in the K-group and a generalization of the Atkinson theorem.
In [**IS6**] we established semi-Fredholm theory in unital C*-algebras as a continuation of the approach by Keckic and Lazovic. We introduced the notion of a semi-Fredholm type element and semi-Weyl type element with respect to the ideal of finite type elements in a unital C*-algebra and obtain a generalization in this setting of several results from the classical semi-Fredholm and semi-Weyl theory of operators on Hilbert spaces.

Definition

[**KL**] Let $a \in \mathcal{A}$ and p, q be projections in $\mathcal{A}$. We say that a is invertible up to pair (p, q) if there exists some $b \in \mathcal{A}$ such that

$$(1-q)a(1-p)b = 1-q, \;\; b(1-q)a(1-p) = 1-p.$$

We refer to such b as almost inverse of a, or (p, q)-inverse of a.

We notice that if b is a (p, q)-inverse of a, then $(1-p)b(1-q)$ is also a (p, q)-inverse of a.

Let $F \in B^a(H_{\mathcal{A}})$. Then F is invertible up to some pair of orthogonal projections (P, Q) if and only if there exists a decomposition

$$H_{\mathcal{A}} = M \oplus M^{\perp} \xrightarrow{F} N \oplus N^{\perp} = H_{\mathcal{A}}$$

with respect to which F has the matrix

$$\begin{bmatrix} F_1 & F_2 \\ F_3 & F_4 \end{bmatrix},$$

where F_1 is an isomorphism, $M = Im(I - P)$ and $N = Im(I - Q)$. It can be shown that this equivalent to the statement that there exists a decomposition

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}}$$

with respect to which F has the matrix

$$\begin{bmatrix} F_1 & 0 \\ 0 & F_4 \end{bmatrix},$$

where F_1 is an isomorphism.

Definition

[**KL**] Let $\mathcal{A}$ be an unital C^*-algebra, and $\mathcal{F} \subseteq \mathcal{A}$ be a subalgebra which satisfies the following conditions:
(i) $\mathcal{F}$ is a selfadjoint ideal in $\mathcal{A}$, i.e. for all $a \in \mathcal{A}, b \in \mathcal{F}$ there holds $ab, ba \in \mathcal{F}$, and $a \in \mathcal{F}$ implies $a^* \in \mathcal{F}$;
(ii) There is an approximate unit p_α in the norm topology for $\mathcal{F}$ consisting of projections.
Such ideal we shall call as the ideal of finite type elements.

Let H be a separable Hilbert space and $\mathcal{K}(H)$ be the ideal of compact operators in $B(H)$. Then $\mathcal{K}(H)$ satisfies the conditions of the above Definition.

Let $\mathcal{M}$ be a Hilbert C^*-module over a unital C^*-algebra $\mathcal{A}$. We set $\mathcal{K}^*(\mathcal{M})$ to be the closure in the norm topology of the linear span of the operators $\theta_{x,y}$, where $x, y \in \mathcal{M}$ and $\theta_{x,y}(z) = x < y, z >$ for all $z \in \mathcal{M}$. In [**MT**] the operators $\theta_{x,y}$ are called elementary operators. The set $\mathcal{K}^*(\mathcal{M})$ is a closed, two sided self-adjoint ideal in the C^*-algebra $B^a(\mathcal{M})$, and satisfies the conditions of the above definition in the case when $\mathcal{M} = H_{\mathcal{A}}$, see [**MT**].

Let $\mathcal{A}$ be a properly infinite von Neumann algebra acting on a Hilbert space H, and let $\mathfrak{m}$ be the norm closure of the set of all $S \in \mathcal{A}$ for which $P_{\overline{ImS}} \in Proj_0(\mathcal{A})$. Then the couple $(\mathcal{A}, \mathfrak{m})$ satisfies the conditions of the above Definition.

Definition

Let $a \in \mathcal{A}$. We say that a is an upper semi-Fredholm element with respect to the ideal $\mathcal{F}$ if a is invertible up to pair of projections (p, q) where $p \in \mathcal{F}$. Similarly, we say that a is a lower semi-Fredholm element with respect to the ideal $\mathcal{F}$, however in this case we assume that $q \in \mathcal{F}$ (and not p). If both p and q belong to $\mathcal{F}$, we say that a is a Fredholm element with respect to the ideal $\mathcal{F}$.

It can be proved that such upper semi-Fredholm elements correspond to elements in $\mathcal{A}$ that are left invertible module the ideal $\mathcal{F}$, such lower semi-Fredholm elements correspond to elements in $\mathcal{A}$ that are right invertible module the ideal $\mathcal{F}$, whereas such Fredholm elements correspond to elements in $\mathcal{A}$ that are invertible module the ideal $\mathcal{F}$.

Theorem

*[**KL**] Semi-$\mathcal{A}$-Fredholm operators on the standard Hilbert module $H_{\mathcal{A}}$ correspond to semi-Fredholm elements in the C^*-algebra $B^a(H_{\mathcal{A}})$ with respect to the ideal $\mathcal{K}^*(H_{\mathcal{A}})$. Moreover, if $\mathcal{A}$ is a properly infinite von Neumann algebra, then abstract Fredholm elements in $\mathcal{A}$ with respect to the ideal $\mathfrak{m}$ are generalized Fredholm operators in the sense of Breuer.*

Corollary

Let $\mathcal{A}$ be a properly infinite von Neumann algebra. Then an operator $T \in \mathcal{A}$ is $\mathcal{A}$-Fredholm in the sense of Breuer if and only if there exist projections $P, Q \in Proj_0(\mathcal{A})$ such that T is invertible up to (P, Q).

Definition

Let $\mathcal{A}$ be a properly infinite von Neumann algebra and $T \in \mathcal{A}$. We say that T is upper semi-$\mathcal{A}$-Fredholm if there exist projections P, Q in $\mathcal{A}$ such that T is invertible up to (P, Q) where $P \in Proj_0(\mathcal{A})$. Similarly we say that T is lower semi-$\mathcal{A}$-Fredholm , however in this case we assume that $Q \in Proj_0(\mathcal{A})$.

Corollary

Let $\mathcal{A}$ be a properly infinite von Neumann algebra and $T \in \mathcal{A}$. Then T is upper (respectively lower) semi-Fredholm type element in $\mathcal{A}$ with respect to $\mathfrak{m}$ if and only if T is upper (respectively lower) semi-$\mathcal{A}$-Fredholm.

Proposition

[**KL**] Let $a \in \mathcal{A}$ be invertible up to (p, q), and also invertible up to (p', q'), where p, q, p', q' are projections in $\mathcal{F}$.Then in $K(\mathcal{F})$ we have $[p] - [q] = [p'] - [q']$.

Definition

[**KL**] Let $\mathcal{F}$ be the ideal of finite type elements. We say that $a \in \mathcal{A}$ is of Fredholm type (or abstract Fredholm element) if there are projections $p, q \in \mathcal{F}$ such that a is invertible up to (p, q). The index of the element a (or abstract index) is the element of the group $K(\mathcal{F})$ defined by

$$\text{ind}(a) = ([p], [q]) \in K(\mathcal{F}),$$

or less formally

$$\text{ind}(a) = [p] - [q].$$

Let $B(\mathcal{A})$ denote the set of all $\mathcal{A}$ - linear bounded adjointable operators on $\mathcal{A}$ when $\mathcal{A}$ is considered as a right Hilbert module over itself. Since $\mathcal{A}$ is self-dual Hilbert module over itself, by [**MT**] all operators that belong to $B(\mathcal{A})$ are adjointable. Moreover, by [**MT**] the set $B(\mathcal{A})$ is a unital C^* -algebra.
Let V be a map from $\mathcal{A}$ into $B(\mathcal{A})$ given by $V(a) = L_a$ for all $a \in \mathcal{A}$ where L_a is the corresponding left multiplier by a. Then V is an isometric *-homomorphism, and, since $\mathcal{A}$ is unital, it follows that V is in fact an isomorphism. Thus, $B(\mathcal{A})$ can be identified with $\mathcal{A}$ by considering the left multipliers.
If $\mathcal{F}$ is an ideal of finite type elements in $\mathcal{A}$, then it is not hard to see that $V(\mathcal{F})$ is an ideal of finite type elements in $B(\mathcal{A})$, so we may identify $\mathcal{F}$ with $V(\mathcal{F})$.

Definition

Let $F \in B(\mathcal{A})$. We say that $F \in \mathcal{MK}\Phi(\mathcal{A})$ if there exists a decomposition

$$\mathcal{A} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = \mathcal{A}$$

with respect to which F has the matrix $\begin{pmatrix} F_1 & 0 \\ 0 & F_4 \end{pmatrix}$ where F_1 is an isomorphism and $P_{N_1}, P_{N_2} \in \mathcal{F}$. We put then

$$indexF = [P_{N_1}] - [P_{N_2}]$$

in $K(\mathcal{F})$.

Notice that since N_1 and N_2 are closed and complementable, one can show that they are orthogonally complementable, hence P_{N_1} and P_{N_2} are well defined.

Moreover, one can show that this approach is equivalent to the approach introduced above and that $\mathcal{MK}\Phi$ -operators correspond to Fredholm type elements in $\mathcal{A}$.

Proposition

[**KL**] The set of Fredholm type elements is open in $\mathcal{A}$ and the index is a locally constant function.

Proposition

[**KL**] a) Let $a \in \mathcal{A}$ be of Fredholm type, and let $f \in \mathcal{F}$. Then $a + f$ is also of Fredholm type, and index $(a + f) =$ index a.

b) If $f \in \mathcal{F}$, then $1 + f$ is of Fredholm type, and index $(1 + f) = 0$. Moreover, there is $p \in \mathcal{F}$ such that $1 + f$ is invertible up to (p, p).

Proposition

[**KL**] a) If a is of Fredholm type, then a is invertible modulo $\mathcal{F}$;

b) Conversely, if a is invertible modulo $\mathcal{F}$, then a is of Fredholm type.

Theorem

*[**KL**] (index theorem). Let $\mathcal{A}$ be a unital C^*-algebra, and let $\mathcal{F} \subseteq \mathcal{A}$ be an algebra of finite type elements. If t_1 and t_2 are Fredholm type elements, then $t_1 t_2$ is of Fredholm type as well. Moreover there holds*

$$\textit{index}\,(t_1 t_2) = \textit{index}\; t_1 + \textit{index}\; t_2.$$

In other words, if we denote the set of all Fredholm type elements by Fred (F), then Fred $(\mathcal{F})$ is a semigroup (with unit) with respect to multiplication, and the mapping index is a homomorphism from $(\text{Fred}\,(\mathcal{F}), \cdot)$ to $(K(\mathcal{F}), +)$.

Lemma

Let $p \in \mathcal{F}$ be a projection. Then the couple

$$((1-p)\mathcal{A}(1-p), (1-p)\mathcal{F}(1-p))$$

satisfies the conditions of the above Definition.

Corollary

Let $a \in \mathcal{A}$ and p be a projection in $\mathcal{F}$. Then a is a Fredholm type element in $\mathcal{A}$ with respect to the ideal $\mathcal{F}$ if and only if $(1-p)a(1-p)$ is a Fredholm type element in $(1-p)\mathcal{A}(1-p)$ with respect to the ideal $(1-p)\mathcal{F}(1-p)$ and in this case index $a =$ index $(1-p)a(1-p)$.

Lemma

Let $a \in \mathcal{A}$. Then a is an upper semi-Fredholm element if and only if a is left invertible up to some projection $p \in \mathcal{F}$. Similarly, a is a lower semi-Fredholm element if and only if a is right invertible up to some projection $q \in \mathcal{F}$.

Corollary

Let $\mathcal{A}$ be a properly infinite von Neumann algebra acting on a Hilbert space H and $T \in \mathcal{A}$. Then T is upper semi-$\mathcal{A}$-Fredholm if and only if there exists some $P \in Proj_0(\mathcal{A})$ such that T is bounded below on $(I-P)(H)$. Similarly, T is lower semi-$\mathcal{A}$-Fredholm if and only if there exists some $Q \in Proj_0(\mathcal{A})$ such that $(I-Q)(H) \subseteq ImT$.

Lemma

*[**IS5**] Let $F \in B^a(M)$ where M is a Hilbert C^*-module and suppose that ImF is closed. Then the following statements hold:*
a) $F \in \mathcal{M}\Phi_+(M)$, if and only if $\ker F$ is finitely generated;
b) $F \in \mathcal{M}\Phi_-(M)$, if and only if $ImF^\perp$ is finitely generated.

Corollary

Let $\mathcal{A}$ be a von Neumann algebra and $T \in \mathcal{A}$. Then the following statements hold.
1) If T is upper semi-$\mathcal{A}$-Fredhoolm, then $P_{\ker T} \in Proj_0(\mathcal{A})$. In particular, if ImT is closed, then T is upper semi-$\mathcal{A}$-Fredholm if and only if $P_{\ker T} \in Proj_0(\mathcal{A})$.
2) If T is lower semi-$\mathcal{A}$-Fredholm, then $P_{\overline{ImT}^\perp} \in Proj_0(\mathcal{A})$. In particular, if ImT is closed, then T is lower semi-$\mathcal{A}$-Fredholm if and only if $P_{ImT^\perp} \in Proj_0(\mathcal{A})$.

Lemma

Let $a \in G(\mathcal{A})$ and suppose that $K(\mathcal{F})$ satisfies the cancellation property i.e. for any pair of projections p, q in $\mathcal{F}$ we have that $p \sim q$ whenever $[p] = [q]$. Then for every $f \in \mathcal{F}$ we have that $a + f$ is left invertible in $\mathcal{A}$ if and only if $a + f$ is right invertible in $\mathcal{A}$.

For $\alpha \in \mathcal{A}$ we may let αI be the operator on $H_{\mathcal{A}}$ given by

$$\alpha I(x_1, x_2, \dots) = (\alpha x_1, \alpha x_2, \dots).$$

It is straightforward to check that αI is an $\mathcal{A}$-linear operator on $H_{\mathcal{A}}$. Moreover, αI is bounded and $\| \alpha I \| = \| \alpha \|$. Finally, αI is adjointable and its adjoint is given by $(\alpha I)^* = \alpha^* I$.
We give then the following generalization of the well known Fredholm alternative.

Corollary

Let $K \in \mathcal{K}^*(H_{\mathcal{A}})$ and $\alpha \in G(\mathcal{A})$. Suppose that $K_0(\mathcal{A})$ satisfies the cancellation property. Then the equation $(K - \alpha I)\ x = y$ has a solution for every $y \in H_{\mathcal{A}}$ if and only if $K - \alpha I$ is bounded below. In this case the solution of the above equation is unique.

Definition

Let p, q be projections in $\mathcal{A}$. We will denote $p \preceq q$ if there exists some projection p' such that $p' \leq q$ and $p \sim p'$.

Definition

Let $a \in \mathcal{A}$. We say that a is an upper semi-Weyl type element with respect to the ideal $\mathcal{F}$ if there exist projections p, q in $\mathcal{A}$ such that $p \in \mathcal{F}$, $p \preceq q$ and a is invertible up to pair (p, q). Similarly we say that a is a lower semi-Weyl type element with respect to the ideal $\mathcal{F}$, only in this case we assume that $q \in \mathcal{F}$ and $q \preceq p$. Finally, we say that a is a Weyl type element with respect to the ideal $\mathcal{F}$ if a is invertible up to pair (p, q) where p, q are projections in $\mathcal{F}$ and $p \sim q$.

Set
$\mathcal{K}\Phi_{+}(\mathcal{A}) = \{a \in \mathcal{A} \mid a$ is upper semi-Fredholm type element $\}$,
$\mathcal{K}\Phi_{-}(\mathcal{A}) = \{a \in \mathcal{A} \mid a$ is lower semi-Fredholm type element $\}$,
$\mathcal{K}\Phi(\mathcal{A}) = \{a \in \mathcal{A} \mid a$ is Fredholm type element $\}$,
$\mathcal{K}\Phi_{+}^{-}(\mathcal{A}) = \{a \in \mathcal{A} \mid a$ is upper semi-Weyl type element $\}$,
$\mathcal{K}\Phi_{-}^{+}(\mathcal{A}) = \{a \in \mathcal{A} \mid a$ is lower semi-Weyl type element $\}$,
$\mathcal{K}\Phi_{0}(\mathcal{A}) = \{a \in \mathcal{A} \mid a$ is Weyl type element $\}$.

Proposition

The sets $\mathcal{K}\Phi_{+}(\mathcal{A})$, $\mathcal{K}\Phi_{-}(\mathcal{A})$, $\mathcal{K}\Phi_{+}^{-}(\mathcal{A})$, $\mathcal{K}\Phi_{-}^{+}(\mathcal{A})$, $\mathcal{K}\Phi_{0}(\mathcal{A})$, $\mathcal{K}\Phi_{+}(\mathcal{A}) \setminus \mathcal{K}\Phi_{+}^{-}(\mathcal{A})$, $\mathcal{K}\Phi_{-}(\mathcal{A}) \setminus \mathcal{K}\Phi_{-}^{+}(\mathcal{A})$ and $\mathcal{K}\Phi(\mathcal{A}) \setminus \mathcal{K}\Phi_{0}(\mathcal{A})$ are open in the norm topology of $\mathcal{A}$.

Corollary

Let $f : [0,1] \to \mathcal{A}$ *be a continuous map such that* $f([0,1]) \subseteq \mathcal{K}\Phi_+(\mathcal{A}) \cup \mathcal{K}\Phi_-(\mathcal{A})$.
Then the following statements hold.
1) If $f(0) \in \mathcal{K}\Phi_+(\mathcal{A}) \setminus \mathcal{K}\Phi(\mathcal{A})$, *then* $f(1) \in \mathcal{K}\Phi_+(\mathcal{A}) \setminus \mathcal{K}\Phi(\mathcal{A})$.
2) If $f(0) \in \mathcal{K}\Phi_-(\mathcal{A}) \setminus \mathcal{K}\Phi(\mathcal{A})$, *then* $f(1) \in \mathcal{K}\Phi_-(\mathcal{A}) \setminus \mathcal{K}\Phi(\mathcal{A})$.
3) If $f(0) \in \mathcal{K}\Phi_+^-(\mathcal{A})$, *then* $f(1) \in \mathcal{K}\Phi_+^-(\mathcal{A})$.
4) If $f(0) \in \mathcal{K}\Phi_-^+(\mathcal{A})$, *then* $f(1) \in \mathcal{K}\Phi_-^+(\mathcal{A})$.
5) If $f(0) \in \mathcal{K}\Phi_0(\mathcal{A})$, *then* $f(1) \in \mathcal{K}\Phi_0(\mathcal{A})$.
6) If $f(0) \in \mathcal{K}\Phi_+(\mathcal{A}) \setminus \mathcal{K}\Phi_+^-(\mathcal{A})$, *then* $f(1) \in \mathcal{K}\Phi_+(\mathcal{A}) \setminus \mathcal{K}\Phi_+^-(\mathcal{A})$.
7) If $f(0) \in \mathcal{K}\Phi_-(\mathcal{A}) \setminus \mathcal{K}\Phi_-^+(\mathcal{A})$, *then* $f(1) \in \mathcal{K}\Phi_-(\mathcal{A}) \setminus \mathcal{K}\Phi_-^+(\mathcal{A})$.
8) If $f(0) \in \mathcal{K}\Phi(\mathcal{A}) \setminus \mathcal{K}\Phi_0(\mathcal{A})$, *then* $f(1) \in \mathcal{K}\Phi(\mathcal{A}) \setminus \mathcal{K}\Phi_0(\mathcal{A})$.
9) If $f(0) \in \mathcal{K}\Phi_+(\mathcal{A})$, *then* $f(1) \in \mathcal{K}\Phi_+(\mathcal{A})$.
10) If $f(0) \in \mathcal{K}\Phi_-(\mathcal{A})$, *then* $f(1) \in \mathcal{K}\Phi_-(\mathcal{A})$.
11) If $f(0) \in \mathcal{K}\Phi(\mathcal{A})$, *then* $f(1) \in \mathcal{K}\Phi(\mathcal{A})$ *and index* $f(0) =$ *index* $f(1)$.

Corollary

Let $a \in \mathcal{A}$. Then the following statements hold.
1) If a belongs to the boundary of $\mathcal{K}\Phi(\mathcal{A})$ in $\mathcal{A}$, then
$a \in \mathcal{A} \setminus \mathcal{K}\Phi_+(\mathcal{A}) \cup \mathcal{K}\Phi_-(\mathcal{A})$.
2) If a belongs to the boundary of $\mathcal{K}\Phi_+^-(\mathcal{A})$ in $\mathcal{A}$, then $a \in \mathcal{A} \setminus \mathcal{K}\Phi_+(\mathcal{A})$.
3) If a belongs to the boundary of $\mathcal{K}\Phi_-^+(\mathcal{A})$ in $\mathcal{A}$, then $a \in \mathcal{A} \setminus \mathcal{K}\Phi_-(\mathcal{A})$.
4) If a belongs to the boundary of $\mathcal{K}\Phi_0(\mathcal{A})$ in $\mathcal{A}$, then $a \in \mathcal{A} \setminus \mathcal{K}\Phi(\mathcal{A})$.

Proposition

Let $a \in \mathcal{A}$. Then the following holds.
1) If $a \in \mathcal{K}\Phi_{+}^{-}(\mathcal{A})$ and $f \in \mathcal{F}$, then $a + f \in \mathcal{K}\Phi_{+}^{-}(\mathcal{A})$.
2) If $a \in \mathcal{K}\Phi_{-}^{+}(\mathcal{A})$ and $f \in \mathcal{F}$, then $a + f \in \mathcal{K}\Phi_{-}^{+}(\mathcal{A})$.
3) If $a \in \mathcal{K}\Phi_{0}(\mathcal{A})$ and $f \in \mathcal{F}$, then $a + f \in \mathcal{K}\Phi_{0}(\mathcal{A})$.

Lemma

Let $a \in \mathcal{K}\Phi_{+}^{-}(\mathcal{A}) \cap \mathcal{K}\Phi_{-}^{+}(\mathcal{A}) \cap \mathcal{K}\Phi(\mathcal{A})$. Then there exist projections p, q in $\mathcal{F}$ such that a is invertible up to (p, q), $qa(1-p) = 0$, $p \preceq q$ and $q \preceq p$.

Proposition

Let $a \in \mathcal{A}$. Then the following statements hold.
1) $a \in \mathcal{K}\Phi_{+}^{-}(\mathcal{A})$ if and only if there exist a left invertible element $b \in \mathcal{A}$ and some $f \in \mathcal{F}$ such that $a = b + f$,
2) $a \in \mathcal{K}\Phi_{-}^{+}(\mathcal{A})$ if and only if there exist a right invertible element $b \in \mathcal{A}$ and some $f \in \mathcal{F}$ such that $a = b + f$,
3) $a \in \mathcal{K}\Phi_{0}(\mathcal{A})$ if and only if there exist an invertible element $b \in \mathcal{A}$ and some $f \in \mathcal{F}$ such that $a = b + f$.

Corollary

The sets $\mathcal{K}\Phi_{+}^{-}(\mathcal{A}), \mathcal{K}\Phi_{-}^{+}(\mathcal{A})$ *and* $\mathcal{K}\Phi_{0}(\mathcal{A})$ *are semigroups under the multiplication.*

Definition

Let $\mathcal{A}$ be a properly infinite von Neumann algebra and $T \in \mathcal{A}$. We say that T is upper semi-$\mathcal{A}$-Weyl if there exist projections P, Q in $\mathcal{A}$ such that T is invertible up to (P, Q) where $P \in Proj_0(\mathcal{A})$ and $P \preceq Q$.
Similarly we say that T lower semi-$\mathcal{A}$-Weyl, however in this case we assume that $Q \in Proj_0(\mathcal{A})$ and $Q \preceq P$.
Finally, if $P \sim Q$, we say that T is $\mathcal{A}$-Weyl.

Corollary

Let $\mathcal{A}$ be a properly infinite von Neumann algebra and $T \in \mathcal{A}$. Then T is upper (respectively lower) semi-Weyl type element in $\mathcal{A}$ with respect to $\mathfrak{m}$ if and only if T is upper (respectively lower) semi-$\mathcal{A}$-Weyl. Finally, T is Weyl type element in $\mathcal{A}$ with respect to $\mathfrak{m}$ if and only if T is $\mathcal{A}$-Weyl.

Corollary

Let $\mathcal{A}$ be a properly infinite von Neumann algebra acting on a Hilbert space H and $T \in \mathcal{A}$. Then the following statements hold.
1) T is upper semi-$\mathcal{A}$- Weyl if and only if there exist some $S \in \mathcal{A}$ and $F \in \mathfrak{m}$ such that S is bounded below and $T = S + F$.
2) T is lower semi-$\mathcal{A}$- Weyl if and only if there exist some $S \in \mathcal{A}$ and $F \in \mathfrak{m}$ such that S is surjective and $T = S + F$.
3) T is $\mathcal{A}$-Weyl if and only if there exist some $S \in \mathcal{A}$ and $F \in \mathfrak{m}$ such that S is invertible and $T = S + F$.

Corollary

Let $\mathcal{A}$ be a properly infinite von Neumann algebra. Then we have that $\mathcal{K}\Phi_{+}^{-}(\mathcal{A}) \cap \mathcal{K}\Phi_{-}^{+}(\mathcal{A}) \cap \mathcal{K}\Phi(\mathcal{A}) = \mathcal{K}\Phi_0(\mathcal{A})$.

Definition

[**IS1**] Let $F \in \mathcal{M}\Phi_{+}(H_{\mathcal{A}})$. We say that $F \in \mathcal{M}\Phi_{+}^{-\prime}(H_{\mathcal{A}})$ if there exists a decomposition

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}}$$

with respect to which

$$F = \begin{bmatrix} F_1 & 0 \\ 0 & F_4 \end{bmatrix},$$

where F_1 is an isomorphism, N_1 is closed, finitely generated and $N_1 \preceq N_2$. Similarly, we define the class $\mathcal{M}\Phi_{-}^{+\prime}(H_{\mathcal{A}})$, only in this case $F \in \mathcal{M}\Phi_{-}(H_{\mathcal{A}})$, N_2 is finitely generated and $N_2 \preceq N_1$.
Such operators we will call semi-$\mathcal{A}$-Weyl operators.
Further, we define $\mathcal{M}\Phi_{0}(H_{\mathcal{A}})$ to be the set of all $F \in \mathcal{M}\Phi(H_{\mathcal{A}})$ for which there exists an $\mathcal{M}\Phi$-decomposition

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}},$$

where $N_1 \cong N_2$. Such operators we will call $\mathcal{A}$-Weyl operators.

Theorem

*[**IS1**] Let $F \in B^a(H_{\mathcal{A}})$. The following statements are equivalent:*
1) $F \in \mathcal{M}\Phi_{+}^{-\prime}(H_{\mathcal{A}})$,
2) There exist $D \in B^a(H_{\mathcal{A}}), K \in \mathcal{K}^*(H_{\mathcal{A}})$ such that D is bounded below and $F = D + K$.

Corollary

*[**IS1**] Let $D \in B^a(H_{\mathcal{A}})$. The following statements are equivalent:*
1) $D \in \mathcal{M}\Phi_{-}^{+\prime}(H_{\mathcal{A}})$,
2) There exist a surjective operator $Q \in B^a(H_{\mathcal{A}})$ and $K \in \mathcal{K}^*(H_{\mathcal{A}})$ such that $D = Q + K$.

Theorem

Let $B^a(H_{\mathcal{A}})$. *Then the following statements are equivalent:*
1) $F \in \mathcal{M}\Phi_0(H_{\mathcal{A}})$,
2) There exist an invertible $D \in B^a(H_{\mathcal{A}})$ *and* $K \in \mathcal{K}^*(H_{\mathcal{A}})$ *such that* $F = D + K$.

Lemma

Let $F \in \mathcal{M}\Phi_+^{-'}(H_{\mathcal{A}}) \cap \mathcal{M}\Phi_-^{+'}(H_{\mathcal{A}})$. *Then there exists an* $\mathcal{M}\Phi$*-decomposition*

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}}$$

for F *with the property that* $N_1 \preceq N_2$ *and* $N_2 \preceq N_1$.

Thank you for attention !

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