Explicit construction of explicit real algebraic functions and real algebraic manifolds via Reeb graphs

Naoki Kitazawa
Institute of Mathematics for Industry, Kyushu University (2023/6–)
/Osaka Central Advanced Mathematical Institute
naokikitazawa.formath@gmail.com

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for example.

Real algebraic functions and maps.

- According to Nash (and Tognoli), a smooth closed manifold is regarded as a non-singular real algebraic manifold which is also a real algebraic set.
- Existence theory of real algebraic manifolds and real algebraic maps has been well-known.
  $\Rightarrow$ We can easily attain such objects and morphisms whereas it is very difficult to know explicitly.
- Some specific examples are well-known (e. g. canonical embeddings and projections of spheres in Euclidean spaces, Morse functions on Lie groups and symmetric spaces etc.).
  $\Rightarrow$ However, it is difficult to know global structures and properties.

Construction of real algebraic functions with information on preimages and important real polynomials.

Problem 1

Can we obtain real algebraic functions (or more generally maps) in constructive ways? Can we know their precise global structures and properties? For example, can we know about preimages? Can we know about the real polynomials for the zero sets and the real algebraic manifolds?

We present some of our explicit answers. We challenge Problem 2 (, which may be only presented in a bit different way).

Problem 2

Can we have real algebraic functions whose preimages satisfy suitably given conditions on some smooth (non-singular) real algebraic manifolds and know information on important real polynomials (for the zero sets etc.)?

A short introductory remark on Problem 2.

Problem 3 (2006 Sharko)

Can we have a smooth nice function whose Reeb graph is isomorphic to a given graph?

⇒ The Reeb graph of a smooth function: a natural graph whose underlying space is the (natural quotient) space of (the manifold) consisting of all components of preimages.
⇒ Some answers: Sharko (2006: functions on closed surfaces), Masumoto-Saeki (2010: an improvement of the result of Sharko for arbitrary finite graphs), Michalak (2018: Morse functions on closed manifolds) .

Problem 4 (2019– K)

Can we have a smooth nice function whose preimages are homeomorphic (diffeomorphic) to prescribed manifolds?

We consider these problems in the real algebraic category.

Notation on fundamental (smooth) manifolds and related notions etc..

- $\mathbb{R}^k$: the k-dim. Euclidian space ($\mathbb{R}^1 := \mathbb{R}$).
  $\Rightarrow$ a simplest smooth manifold, a Riemannian manifold endowed with the standard Euclidean metric and a non-singular real algebraic manifold (the k-dim. real affine space).
- $||p||$: the distance between $p \in \mathbb{R}^k$ and the origin 0.
- S^k (D^{k+1}):=$\{p \in \mathbb{R}^{k+1} \mid ||x|| =(\text{resp. } \leq)1\}$: the k-dim. unit sphere (resp. $(k+1)$-dim. unit disk).
- X^l : an l-dim. manifold X ("l" in "X^l" is for the dim.).

Smooth maps and diffeomorphisms.

- $\pi_{k,k'} : \mathbb{R}^k \to \mathbb{R}^{k'}$ $(k > k' \geq 1)$: the map defined by $\pi_{k,k'}(x_1, x_2) := x_1$ for $(x_1, x_2) \in \mathbb{R}^k = \mathbb{R}^{k'} \times \mathbb{R}^{k-k'}$ and it is a canonical projection (and the restriction to S^{k-1} is a canonical projection of the unit sphere).
- $f : M^m \to N^n$: a smooth map. $p \in M^m$ is a singular point of f : at p the rank of the differential df_p is smaller than $\min\{m, n\}$. $f(p)$ is a singular value of f. $S(f)$ denotes the set of all singular points of f.
- $f : M^m \to \mathbb{R}$ is a Morse function : f is smooth and at each singular point p, identified with $x = 0 \in \mathbb{R}^m$, it is represented as a map of the form

$$(x_1, \cdots, x_m) \mapsto \Sigma_{j=1}^{m-i(p)} {x_j}^2 - \Sigma_{j=1}^{i(p)} {x_{m-i(p)+j}}^2 + f(p)$$

for some integer $0 \leq i(p) \leq m$.
- Two smooth manifolds are diffeomorphic : $\exists$ a diffeomorphism, or a smooth homeomorphism with no singular points from a manifold to the other.

An algebraic domain.

Definition 1

$k \geq 2$: an integer.
An algebraic domain $R^k \subset \mathbb{R}^k$ is a bounded open set s. t.

- The closure $\overline{R}$ is surrounded by $\sqcup_j S_j$: $\{S_j\}_{j=1}^{l} \subset \overline{R} - R$ is a family of finitely many mutually disjoint connected non-singular real algebraic hypersurfaces.
- S_j: a component of the zero set $\{x \mid f_j(x) = 0\}$.
- $1 \leq \forall j \leq l$, $(\{x \in \mathbb{R}^k \mid f_j(x) = 0\} - S_j) \bigcap \overline{U_R} = \emptyset$ where $\overline{U_R}$ is the closure of a sufficiently small open neighborhood U_R of $\overline{R}$.

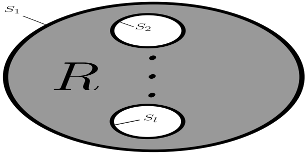

Figure: An algebraic domain $R^2 \subset \mathbb{R}^2$ surrounded by circles: "dots" are for abbreviation.

The Reeb graph of a smooth function.

For a smooth function $f : M \to \mathbb{R}$, we consider the natural quotient space W_f of M consisting of all components of preimages.

Theorem 1 (2020: Saeki)

Let M be a closed manifold here. If the set $f(S(f))$ of all singular values of f is finite, then W_f is a graph whose vertex set consists of all components containing some singular points of f.

W_f is the Reeb graph of f and $q_f : M \to W_f$ is the quotient map.

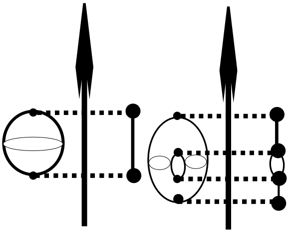

Figure: The Reeb graph $W_{\pi_{m,1}|_{S^{m-1}}}$ of the projection $\pi_{m,1}|_{S^{m-1}}$ and that of a function on a torus obtained by a natural height: they are also Morse.

The Poincaré-Reeb graph of an algebraic domain.

We introduce a similar and different notion.

Definition 2

The Poincaré-Reeb graph G_R of an algebraic domain $R^k \subset \mathbb{R}^k$ is the graph as follows.

- G_R is the natural quotient space of $\overline{R}$ consisting of all components of preimages of the function $\pi_{k,1}|_{\overline{R}}$.
- The vertex set of G_R consists of all components containing some singular points of the function $\pi_{k,1}|_{\overline{R}-R}$.

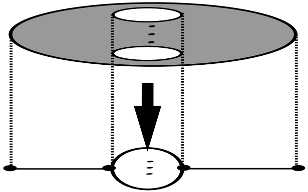

Figure: The Poincaré-Reeb graph of a domain $R \subset \mathbb{R}^2$ surrounded by circles: "dots" are for abbreviation.

A main theorem—an answer to Problem 3 in the real algebraic situation.

Theorem 2 (2023: K)

$m \geq k \geq 1$: integers.
An algebraic domain $R^k \subset \mathbb{R}^k$ is represented as $R = U_R \bigcap \bigcap_{j=1}^{l} \{x \mid f_j(x) > 0\}$ where we abuse the notation in Definition 1.
$\rightarrow$ For the Poincaré-Reeb graph G_R of R, $\exists$ a non-singular closed real algebraic manifold M^m, $\exists$ a smooth real algebraic function $f : M \rightarrow \mathbb{R}$ whose Reeb graph W_f is isomorphic to G_R.

$\rightarrow$ The figure in the previous slide etc. shows a simplest example.

Proof of Theorem 2.

STEP 1 $S := \{(x,y) \in \overline{R} \times \mathbb{R}^{m-k+1} \mid \prod_{j=1}^{l}(f_j(x)) - ||y||^2 = 0\}$.

STEP 2 We see that this is a union of finitely many components of the zero set of the real polynomial and a **non-singular** closed real algebraic manifold.
$\Rightarrow$ We apply implicit function theorem for this.

STEP 3 Put $M := S$ and $f := \pi_{k,1} \circ \pi_{m+1,k}|_M = \pi_{m+1,1}|_M$.

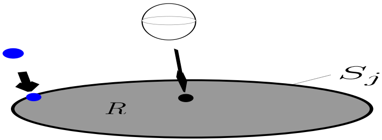

Figure: Preimages of some points of $R^k \subset \mathbb{R}^k$ (for $\pi_{m+1,k}|_M$).

$\rightarrow$ In the case $R = \mathrm{Int}\ D^k$, we have $\pi_{m+1,k}|_{S^m}$.

Remarks on the proof of Theorem 2.

Remark 1

In general, **topologically**, we have so-called special generic maps as the maps onto $\overline{R} \subset \mathbb{R}^k$.

$\Rightarrow$ Special generic maps are generalizations of **Morse functions with exactly two singular points on spheres** (Morse functions in **Reeb's theorem**) and **canonical projections of unit spheres**.
$\Rightarrow$ They are well-known to be nice classes of generic smooth maps in knowing the topologies and the differentiable structures of the manifolds (of the domains) precisely.
$\Rightarrow$ By the definition (the structures of) the manifolds and the maps are strongly restricted and some manifolds regarded as elementary in some senses (e.g. **connected sums** $\sharp(S^{i_j} \times S^{m-i_j-1})$ **of finitely many manifolds diffeomorphic to the products** $S^{i_j} \times S^{m-i_j-1}$ $\mathbf{(}1 \leq i_j \leq m-2\mathbf{)}$) admit such maps in considerable cases (1960s Calabi, 1990s– Saeki, Sakuma, 2010s– Nishioka, Wrazidlo, 2020s– K etc.).

Additional remarks on our arguments.

Remark 2

According to a study by A. Bodin, P. Popescu-Pampu and M. S. Sorea in 2022, for algebraic domains in $\mathbb{R}^2$ and Poincaré-Reeb graphs of a certain wide class, we have situations as presented in Theorem 2.

$\Rightarrow$ We cannot know the real polynomials for the curves surrounding the domains explicitly in general.

Remark 3

G: a so-called <u>generic</u> graph. $\rightarrow$ We have a Morse function f s.t.

- At distinct singular points, the values are distinct.
- The Reeb graph W_f is isomorphic to G.

A suitable approximation presents a smooth real algebraic function. However, we cannot apply this for graphs which are not generic.

$\Rightarrow$ On the other hands, we avoid approximations and existence.

An answer to Problem 4 in the real algebraic category.

Theorem 3 (2023K)

$l > 3$, $m > 2$: integers. $\{t_j\}_{j=1}^{l} \subset \mathbb{R}$: an increasing sequence.
$\{F_j\}_{j=1}^{l-1}$: a family of smooth closed manifolds s.t.

- $F_1 = F_{l-1} = S^{m-1}$.
- *The others are either S^{m-1} or represented as $\sharp(S^{i_j} \times S^{m-i_j-1})$* **for some integers** $1 \leq i_j \leq m-2$: **the connected sum is taken in the smooth category**.
- *For adjacent integers $1 \leq j \leq l-2$ and $j+1$, either F_j or F_{j+1} is not diffeomorphic to the unit sphere.*

$\rightarrow \exists$ a non-singular real algebraic closed and connected manifold M^m, $\exists$ a smooth real algebraic function $f : M \rightarrow \mathbb{R}$ and the following are enjoyed.

1. *$S(f)$ is finite and $\{t_j\}_{j=1}^{l}$ is the set of all singular values.*
2. *The preimage $f^{-1}(p_j)$ is diffeomorphic to F_j for $p_j \in (t_j, t_{j+1})$.*

Comments on Theorem 3 (and Theorem 1).

- Main ingredients of our proof of Theorem 3 are as follows.
  - Using maps onto the algebraic domains $R^k \subset \mathbb{R}^k$ in the proof of Theorem 2 inductively by setting $m := k$.
  - More precisely, we obtain a suitable connected component S of the zero set of some real polynomial and a suitable bounded closed region $R = R^{k+1} := {R_S}^{k+1} \subset \mathbb{R}^{k+1}$ s.t.

$$R_S := \{(x,y) \in \overline{R} \times \mathbb{R}^{m-k+1} \mid \prod_{j=1}^{l}(f_j(x)) - ||y||^2 \geq 0\} -$$

$$\text{Int (Finitely many mutually disjoint disks of fixed radii)}$$

$$\partial R = S \sqcup \text{(Finitely many mutually disjoint spheres of fixed radii)}$$

    one after another. We start this from a disk of a suitable fixed radius in $\mathbb{R}^3$.
  - Last we compose the canonical projection $\pi_{m,1}$.
- The construction yields M^m diffeomorphic to S^m or represented as $\sharp(S^{i_j} \times S^{m-i_j})$.

A previous result related to Theorem 3.

Theorem 4 (2021–: K, 2018: Michalak (in the case F_j is a sphere))

$m > 2$: an integer. $G := (V, E)$: *a finite connected graph.*
$\exists g : G \to \mathbb{R}$: *continuous,* $g|_e$ *is injective for each edge* $e \in E$.
$\{F_e\}_{e \in E}$: *a family of smooth closed manifolds labeled by each edge* $e \in E$ *s.t.*

- F_e *is either* S^{m-1} *or* $\sharp(S^{i_j} \times S^{m-i_j-1})$ *for some integers* $1 \leq i_j \leq m-2$ *as in Theorem 3.*
- *If* $e \in E$ *contains a vertex* $v \in V$ *s.t.* $g(v)$ *is a local extremum, then* F_e *is a unit sphere.*
- *If at* $v \in V$ $g(v)$ *is a local extremum, then* v *is of degree* 1.

$\to \exists$ *a closed and connected manifold* M^m, $\exists$ *a Morse function* $f : M \to \mathbb{R}$ *s.t.* W_f *is suitably identified with* G *and the preimage* $q_f(p)$ $(p \in \mathrm{Int}\ e)$ *is diffeomorphic to* F_e.

$\Rightarrow$ In fact the manifolds F_e are more generalized there (by K).

Future problems.

Problem 5

Can we extend our answers to Problems or results to graphs of wider classes and cases conditions on preimages are more general?

⇒ It seems to be difficult due to the fact that the category is very strict (**in the smooth category some results have been obtained as Theorem 4 shows for example**).

Problem 6

For a graph, what is the simplest or most natural smooth function whose Reeb graph is the graph and the manifold? We (may) also need to pose suitable constraints on singularities of functions, preimages, and the category we consider.

⇒ A kind of natural problems.
⇒ For a graph with exactly one edge and two vertices, the canonical projection of a unit sphere seems to be an answer.

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THANK YOU !