Lie structures of the Sheffer group over a Hilbert space

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Umbral Calculus

Umbral calculus is essentially the theory dealing with Sheffer polynomial sequences, which are characterised by the exponential form of their generating function.

The class of Sheffer sequences includes the polynomial sequences of binomial type and Appell sequences.

A central object of studies of umbral calculus is the umbral composition, which equips the set of all Sheffer sequences with a group structure. This group is isomorphic to the Riordan group of infinite lower triangular matrices.

The paper

Cheon, G.-S., Luzón, A., Morón, M.A. Prieto-Martinez, L.F., Song, M.: Finite and infinite dimensional Lie group structures on Riordan groups. *Adv. Math.* 319 (2017), 522–566

introduced a Lie group structure on the Riordan group and discussed the corresponding Lie algebra.

See also

Babenko, I.K.: Algebra, geometry and topology of the substitution group of formal power series. Russian Math. Surveys 68 (2013), 1–68

Multivariate and infinite-dimensional umbral calculus

A lot of research has been done to extend the classical umbral calculus to the multivariate case. However, that research had a significant drawback of being basis-dependent.

The paper

Finkelshtein, D., Kondratiev, Y., Lytvynov, E., Oliveira, M.J.: An infinite dimensional umbral calculus. *J. Funct. Anal.* 276 (2019), 3714–3766

developed foundations of infinite-dimensional, basis-independent umbral calculus.

Sheffer polynomials over a Hilbert space

Let

$$\mathcal{H}_+ \subset \mathcal{H}_0 \subset \mathcal{H}_-$$

be a standard triple of real separable Hilbert spaces, i.e., the Hilbert space $\mathcal{H}_+$ is densely and continuously embedded into $\mathcal{H}_0$ and $\mathcal{H}_-$ is the dual of $\mathcal{H}_+$, while the dual paring between elements of $\mathcal{H}_-$ and $\mathcal{H}_+$ is determined by the inner product in $\mathcal{H}_0$.

Then, for each n, we also get a standard triple

$$\mathcal{H}_+^{\odot n} \subset \mathcal{H}_0^{\odot n} \subset \mathcal{H}_0^{\odot n}.$$

Here $\odot$ denotes the symmetric tensor product. (For a real Hilbert space $\mathcal{H}$, we define $\mathcal{H}^{\odot 0} := \mathbb{R}$.)

For $F_n \in \mathcal{H}_-^{\odot n}$ and $f_n \in \mathcal{H}_+^{\odot n}$, we denote by $\langle F_n, f_n \rangle$ the dual pairing between F_n and f_n.

A *polynomial on* $\mathcal{H}_-$ is a function $p : \mathcal{H}_- \to \mathbb{R}$ of the form

$$p(\omega) = \sum_{k=0}^{n} \langle \omega^{\odot k}, f_k \rangle, \quad \omega \in \mathcal{H}_-,\ f_k \in \mathcal{H}_+^{\odot i}.$$

We denote by $\mathcal{P}(\mathcal{H}_-)$ the vector space of all polynomials on $\mathcal{H}_-$.

By identifying the polynomial $p(\omega)$ with the sequence (f_k), we endow $\mathcal{P}(\mathcal{H}_-)$ with the topology of the topological direct sum of the Hilbert spaces $\mathcal{H}_+^{\odot k}$, $k \in \mathbb{N}_0$.

A *monic polynomial sequence on $\mathcal{H}_-$* is a continuous linear map $P \in \mathcal{L}(\mathcal{P}(\mathcal{H}_-))$ that satisfies

$$P\langle \omega^{\odot n}, f_n\rangle = \sum_{k=0}^{n} \langle \omega^{\odot k}, p_{kn} f_n\rangle, \tag{1}$$

where $p_{kn} \in \mathcal{L}(\mathcal{H}_+^{\odot n}, \mathcal{H}_+^{\odot k})$ and $p_{nn} = \mathbf{1}$.

We identify P with the infinite matrix

$$P = \left[p_{kn}\right]_{k,n\in\mathbb{N}_0},$$

where $p_{kn} = 0$ for $k > n$. Thus, the matrix P is upper-triangular with the identity operators $p_{nn} = \mathbf{1}$ on the diagonal.

Let $p_{kn}^* \in \mathcal{L}(\mathcal{H}_-^{\odot k}, \mathcal{H}_-^{\odot n})$. Then

$$P\langle \omega^{\odot n}, f_n\rangle = \langle p_n(\omega), f_n\rangle,$$

where $p_n(\omega) \in \mathcal{H}_-^{\odot n}$ is given by

$$p_n(\omega) := \sum_{k=0}^{n} p_{kn}^* \omega^{\odot k} \qquad \text{with } p_{nn}^* = \mathbf{1}.$$

Thus, $(p_n(\omega))_{n=0}^{\infty}$ is another form of representation of P.

A monic polynomial sequence $(s_n)_{n=0}^{\infty}$ is called a *Sheffer sequence* if it has the exponential generating function of the form

$$\sum_{n=0}^{\infty} \frac{1}{n!} \langle s_n(\omega), \xi^{\odot n} \rangle = \exp\left[\langle \omega, B(\xi) \rangle\right] A(\xi), \quad \omega \in \mathcal{H}_-, \ \xi \in \mathcal{H}_+, \quad (2)$$

where

$$B(\xi) = \sum_{k=1}^{\infty} b_k \xi^{\odot k}$$

with $b_k \in \mathcal{L}(\mathcal{H}_+^{\odot k}, \mathcal{H}_+)$ and $b_1 = \mathbf{1}$, and

$$A(\xi) = \sum_{k=0}^{\infty} a_k \xi^{\odot k}$$

with $a_k \in \mathcal{L}(\mathcal{H}_+^{\odot k}, \mathbb{R})$, $a_0 = 1$.

The equality (2) is understood as the equality of formal tensor power series in ξ: replace ξ with $t\xi$, where $t \in \mathbb{R}$, and equate the coefficients by each t^k.

We denote by $\mathbb{S}$ the set of all Sheffer sequences.

We denote by $\mathbb{A}$ the set of all *Appell sequences*, i.e., the Sheffer sequences for which $B(\xi) = \xi$:

$$\sum_{n=0}^{\infty} \frac{1}{n!} \langle s_n(\omega), \xi^{\odot n} \rangle = \exp\big[\langle \omega, \xi \rangle\big] A(\xi).$$

We denote by $\mathbb{B}$ the set of all *binomial sequences*, i.e., the Sheffer sequences for which $A(\xi) = 1$:

$$\sum_{n=0}^{\infty} \frac{1}{n!} \langle s_n(\omega), \xi^{\odot n} \rangle = \exp\big[\langle \omega, B(\xi) \rangle\big].$$

Such a sequence satisfies, for all $\omega, \zeta \in \mathcal{H}_-$,

$$s_n(\omega + \zeta) = \sum_{k=0}^{n} \binom{n}{k} s_k(\omega) \odot s_{n-k}(\zeta).$$

Since each monic polynomial sequence $(p_n)_{n=0}^{\infty}$ is identified with an operator $P \in \mathcal{L}(\mathcal{P}(\mathcal{H}_-))$, one can take a product of two such operators and construct a new monic polynomial sequence. Such a product is called an *umbral composition of monic polynoial sequences*.

Explicitly, if

$$P^{(1)} = \left[p_{kn}^{(1)}\right]_{k,n\in\mathbb{N}_0}, \quad P^{(2)} = \left[p_{kn}^{(12}\right]_{k,n\in\mathbb{N}_0},$$

then the umbral product is just the product of the matrices:

$$P = P^{(1)}P^{(2)},$$

i.e.,

$$p_{kn} = \sum_{i=k}^{n} p_{ki}^{(1)} p_{in}^{(2)} .$$

Equipped with the umbral composition, the set of all monic polynomial sequences, $\mathbb{M}$, is a group.

Theorem

(i) $\mathbb{S}$ *is a group under the umbral composition of monic polynomial sequences. Both* $\mathbb{A}$ *and* $\mathbb{B}$ *are subgroups of* $\mathbb{S}$*. Furthermore,* $\mathbb{A}$ *is a commutative, normal subgroup of* $\mathbb{S}$*, and*

$$\mathbb{S} = \mathbb{A} \rtimes \mathbb{B},$$

i.e., the Sheffer group $\mathbb{S}$ *is the semidirect product of the Appell group* $\mathbb{A}$ *and the binomial group* $\mathbb{B}$*.*

Theorem (continuation)

(ii) Let $S^{(1)}$ and $S^{(2)}$ be two Sheffer sequences with the generating functions

$$\sum_{n=0}^{\infty} \frac{1}{n!} \langle s_n^{(i)}(\omega), \xi^{\odot n} \rangle = \exp\left[\langle \omega, B^{(i)}(\xi) \rangle\right] A^{(i)}(\xi).$$

Let $S = S^{(1)} S^{(2)}$. Then the generating function of the Sheffer sequence S is of the form

$$\sum_{n=0}^{\infty} \frac{1}{n!} \langle s_n(\omega), \xi^{\odot n} \rangle = \exp\left[\langle \omega, B(\xi) \rangle\right] A(\xi),$$

where

$$B(\xi) = B^{(1)}(B^{(2)}(\xi))$$

and

$$A(\xi) = A^{(1)}(B^{(2)}(\xi)) \cdot A^{(2)}(\xi).$$

Our aim is to understand $\mathbb{S}$, $\mathbb{A}$ and $\mathbb{B}$ as infinite-dimensional Lie groups,

Lie group of all monic polynomial sequences

Let $\mathbb{V}$ be the vector space of all continuous linear operators $V \in \mathcal{L}(\mathcal{P}(\mathcal{H}_-))$,

$$V = \big[V_{kn}\big]_{k,n\in\mathbb{N}_0}, \quad V_{kn} \in \mathcal{L}(\mathcal{H}_+^{\odot n}, \mathcal{H}_+^{\odot k}), \qquad V_{kn} = 0 \text{ if } k \geq n.$$

We endow $\mathbb{V}$ with the product topology of

$$\prod_{0\leq k<n<\infty} \mathcal{L}(\mathcal{H}_+^{\odot n}, \mathcal{H}_+^{\odot k}).$$

Then $\mathbb{V}$ becomes a complete, locally convex topological vector space.

We have a bijective map

$$\mathbb{M} \ni P \mapsto P - \mathbf{1} \in \mathbb{V}.$$

This map endows $\mathbb{M}$ with a topology and makes $\mathbb{M}$ an infinite dimensional manifold with a global parametrisation.

Proposition

$\mathbb{M}$ is an infinite-dimensional Lie group, $\mathbb{V}$ is its Lie algebra and the Lie bracket on $\mathbb{V}$ is given by

$$[V_1, V_2] = V_1 V_2 - V_2 V_1.$$

For each, $V \in \mathbb{V}$, we have

$$\exp[V] = \sum_{n=0}^{\infty} \frac{1}{n!} V^n \in \mathbb{M},$$

and for each $P \in \mathbb{M}$,

$$V = \log(P) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (P - \mathbf{1})^n \in \mathbb{V}.$$

The map

$$\mathbb{M} \ni P \mapsto \log(P) \in \mathbb{V}$$

gives an equivalent coordinate system on $\mathbb{M}$.

Differentiation of polynomial on $\mathcal{H}_-$

For a function $f : \mathcal{H}_- \to \mathbb{R}$ and $\zeta \in \mathcal{H}_-$, we denote by $D_\zeta f$ the Gateaux derivative of f in direction ζ:

$$D_\zeta f(\omega) = \frac{d}{dt}\Big|_{t=0} f(\omega + t\zeta).$$

Then, for $\xi \in \mathcal{H}_+$,

$$D_\zeta^k \langle \omega^{\odot n}, \xi^{\odot n} \rangle = (n)_k \langle \zeta^{\odot k}, \xi^{\odot k} \rangle \langle \omega^{\odot (n-k)}, \xi^{\odot (n-k)} \rangle,$$

where

$$(n)_k = n(n-1) \cdots (n-k+1).$$

We define a continuous linear operator

$$\nabla^k : \mathcal{P}(\mathcal{H}_-) \to \mathcal{H}_+^{\odot k} \otimes \mathcal{P}(\mathcal{H}_-)$$

satisfying

$$D_\zeta^k p(\omega) = \langle \zeta^{\odot k}, \nabla^k p(\omega) \rangle.$$

Then

$$\nabla^k \langle \omega^{\odot n}, \xi^{\odot n} \rangle = (n)_k \, \xi^{\odot k} \langle \omega^{\odot (n-k)}, \xi^{\odot (n-k)} \rangle,$$

Assume $a_k \in \mathcal{L}(\mathcal{H}_+^{\odot k}, \mathbb{R})$. Then,

$$a_k \nabla^k \in \mathcal{L}(\mathcal{P}(\mathcal{H}_-)).$$

For example,

$$a_k \nabla^k \langle \omega^{\odot n}, \xi^{\odot n} \rangle = (n)_k (a_k \xi^{\odot k}) \langle \omega^{\odot (n-k)}, \xi^{\odot (n-k)} \rangle.$$

Similarly, if $b_k \in \mathcal{L}(\mathcal{H}_+^{\odot k}, \mathcal{H}_+)$,

$$\langle \omega, b_k \nabla^k \rangle \in \mathcal{L}(\mathcal{P}(\mathcal{H}_-)).$$

For example,

$$\langle \omega, b_k \nabla^k \rangle \langle \omega^{\odot n}, \xi^{\odot n} \rangle = (n)_k \langle \omega^{\odot (n-k+1)}, (b_k \xi^{\odot k}) \odot \xi^{\odot (n-k)} \rangle.$$

Theorem

(i) $\mathbb{S}$, $\mathbb{A}$, $\mathbb{B}$ *are Lie subgroup of* $\mathbb{M}$.
(ii) Denote

$$\mathfrak{s} = \{V \in \mathbb{V} \mid \exp(V) \in \mathbb{S}\},$$
$$\mathfrak{a} = \{V \in \mathbb{V} \mid \exp(V) \in \mathbb{A}\},$$
$$\mathfrak{b} = \{V \in \mathbb{V} \mid \exp(V) \in \mathbb{B}\}.$$

Then $\mathfrak{s}$, $\mathfrak{a}$, $\mathfrak{b}$ *are the Lie algebras of* $\mathbb{S}$, $\mathbb{A}$, *and* $\mathbb{B}$, *respectively. The* $\mathfrak{s}$, $\mathfrak{a}$, $\mathfrak{b}$ *are closed subspaces of* $\mathbb{V}$ *and*

$$\mathfrak{a} = \left\{ \sum_{k=1}^{\infty} a_k \nabla^k \mid a_k \in \mathcal{L}(\mathcal{H}_+^{\odot k}, \mathbb{R}) \right\},$$
$$\mathfrak{b} = \left\{ \sum_{k=2}^{\infty} \langle \omega, b_k \nabla^k \rangle \mid b_k \in \mathcal{L}(\mathcal{H}_+^{\odot k}, \mathcal{H}_+) \right\},$$
$$\mathfrak{s} = \mathrm{l.\,s.}(\mathfrak{a} \cup \mathfrak{b}).$$

For the operators $a_k \in \mathcal{L}(\mathcal{H}_+^{\odot k}, \mathbb{R})$ $(k \geq 1)$ and $b_k \in \mathcal{L}(\mathcal{H}_+^{\odot k}, \mathcal{H}_+)$ $(k \geq 2)$, we denote the formal tensor power series

$$a(\xi) = \sum_{k=1}^{\infty} a_k \xi^{\odot k}, \quad b(\xi) = \sum_{k=2}^{\infty} b_k \xi^{\odot k}.$$

Then we will write elements of the Lie algebra $\mathfrak{a}$ as $a(\nabla)$, and elements of the Lie algebra $\mathfrak{b}$ as $\langle \omega, b(\nabla) \rangle$.

Note that, for $\varphi \in \mathcal{H}_+$, the

$$D_\varphi a(\xi) = \sum_{k=1}^{\infty} k\, a_k \big(\xi^{\odot (k-1)} \odot \varphi\big),$$

$$D_\varphi b(\xi) = \sum_{k=2}^{\infty} k\, b_k \big(\xi^{\odot (k-1)} \odot \varphi\big).$$

Theorem

(i) For any $a^{(1)}(\nabla), a^{(2)}(\nabla) \in \mathfrak{a}$,

$$[a^{(1)}(\nabla), a^{(2)}(\nabla)] = 0.$$

(ii) For any $\langle \omega, b^{(1)}(\nabla) \rangle, \langle \omega, b^{(2)}(\nabla) \rangle \in \mathfrak{b}$, *we have*

$$\left[\langle \omega, b^{(1)}(\nabla) \rangle, \langle \omega, b^{(2)}(\nabla) \rangle\right] = \Big\langle \omega, \big(D_{b^{(2)}} b^{(1)}\big)(\nabla) - \big(D_{b^{(1)}} b^{(2)}\big)(\nabla) \Big\rangle.$$

(iii) For any $a(\nabla) \in \mathfrak{a}$ *and* $\langle \omega, b(\nabla) \rangle \in \mathfrak{b}$,

$$\left[a(\nabla), \langle \omega, b(\nabla) \rangle\right] = \big(D_b a\big)(\nabla).$$