Convex bodies of constant width with exponential illumination number

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Smallest known n with $f(n) > n + 1$ is $n = 64$.

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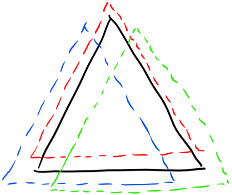

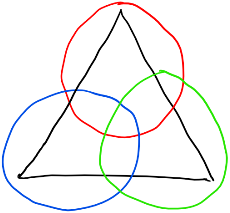

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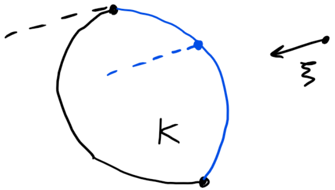
Bourgain and Lindenstrauss (1989): $1.0645^n \le g(n) \le \left(\sqrt{\frac{3}{2}} + o(1)\right)^n$.

Illumination and covering

Let K be a convex body in $\mathbb{E}^n$. A point $x \in \partial K$ is illuminated by a direction $\xi \in \mathbb{S}^{n-1}$ if the ray $\{x + \xi t : t \geq 0\}$ intersects $\mathrm{int}(K)$.



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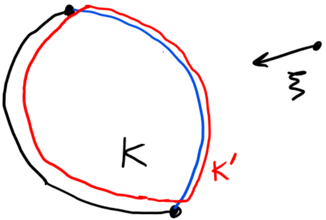

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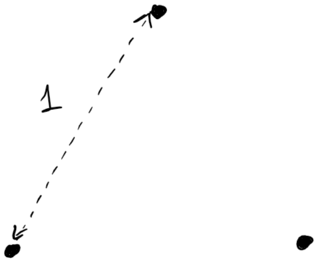

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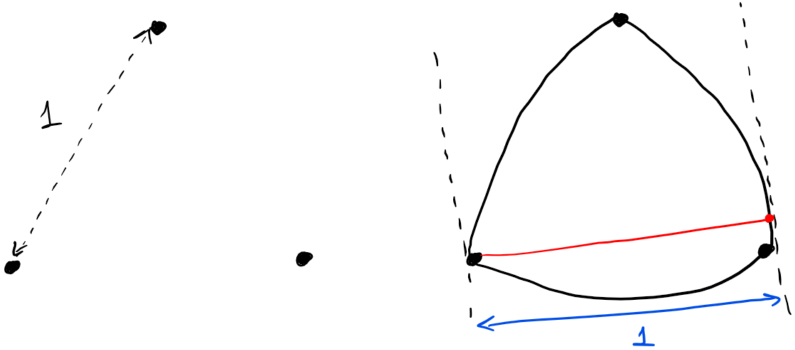

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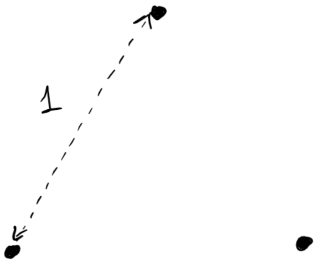

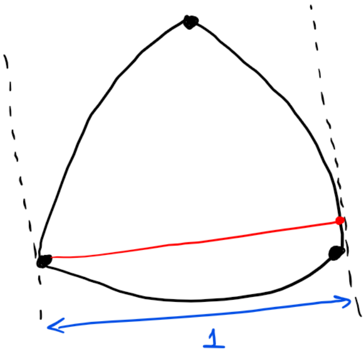

Therefore, it suffices to consider bodies of constant width to compute the Borsuk's number $f(n)$.

Schramm's upper bound on Borsuk's number

Define

$$h(n) := \sup\{h(K) = I(K) : K \text{ is a convex body of constant width in } \mathbb{E}^n\}.$$

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Kalai (2015) asked: does there exist $C > 1$ with $h(n) \geq C^n$ for large n?

Main result

We answer the question of Kalai in the affirmative.

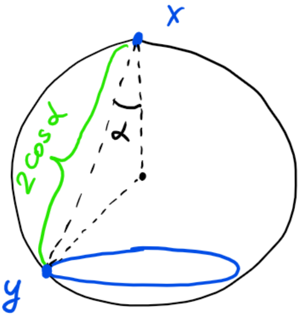
Theorem

$h(n) \geq (\cos(\pi/14) + o(1))^{-n}$.

Main geometric ingredient

For fixed $x \in \mathbb{S}^{n-1}$ and $0 < \alpha \leq \pi/6$ define

$$Q(x, \alpha) := \{x\} \cup \{y \in \mathbb{S}^{n-1} : \|x - y\| = 2\cos\alpha\}.$$



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$$\theta(x, y) := \arccos\left(\frac{x \cdot y}{\|x\|\|y\|}\right).$$

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Lemma

Suppose $0 < \alpha \leq \pi/6$, *K is a convex body in* $\mathbb{E}^n$ *s.t.* $\operatorname{diam} K = 2\cos\alpha$ *and for some* $x \in \mathbb{S}^{n-1}$ *we have* $Q(x, \alpha) \subset K$. *Then* $x \in \partial K$ *and any direction* $\xi \in \mathbb{S}^{d-1}$ *illuminating x satisfies* $\xi \in C(-x, \frac{\pi}{2} - \alpha)$.

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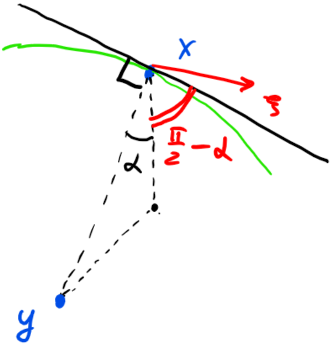

Separation required to control the diameter

For a finite $X \subset \mathbb{S}^{n-1}$, let $\mathcal{W}(X) := \bigcup_{x \in X} Q(x, \alpha)$.

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Suppose $0 < \alpha \leq \pi/6$ *and* $X \subset \mathbb{S}^{n-1}$.

(i) *If* $\theta(x, y) \leq \pi - 2\alpha$ *for all* $x, y \in X$, *then* $\operatorname{diam} X \leq 2\cos\alpha$.

(ii) *If* $4\alpha \leq \theta(x, y) \leq \pi - 6\alpha$ *for all distinct* $x, y \in X$, *then* $\operatorname{diam} \mathcal{W}(X) \leq 2\cos\alpha$.

Main probabilistic lemma

Lemma

Suppose $0 < \psi < \varphi < \frac{\pi}{2}$ *are fixed. Then for every positive integer n there exists a collection* $X = \{x_1, \ldots, x_N\} \subset \mathbb{S}^{n-1}$ *with* $N = \left(\frac{1+o(1)}{\sin\varphi}\right)^n$ *such that*

(a) $\psi \le \theta(x_i, x_j) \le \pi - \psi$ *for all* $1 \le i < j \le N$.

(b) *every point of* $\mathbb{S}^{n-1}$ *is contained in at most* $O(n \log n)$ *spherical caps* $C(x_i, \varphi)$, $1 \le i \le N$.

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Proof: with appropriately selected $N' \approx N$, sample N' uniformly i.i.d. points from $\mathbb{S}^{n-1}$. By Böröczky and Wintsche (2003), which is the adaptation of the ideas of Erdős and Rogers (1961/62) to $\mathbb{S}^{n-1}$, the resulting set satisfies (b) with high probability.

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The expected number of pairs (i, j) not satisfying (a) can be shown to be at most $N'/4$, thus a point from each such pair can be removed to obtain the desired X.

Proof of the main result

Theorem

$h(n) \geq (\cos(\pi/14) + o(1))^{-n}$.

Proof: use the probabilistic lemma with $\varphi = \frac{6\pi}{14} + \epsilon$, $\psi = \frac{6\pi}{14}$, where $\epsilon > 0$.

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Glazyrin ($\geq$2023) noted that our bound $h(n) \geq 1.026^n$ can be improved to $h(n) \geq 1.047^n$ by a slight modification of the construction: choosing the bases of the cones from a concentric sphere of smaller radius.

New lower bound on $g(n)$

Recall that $g(n)$ is the smallest number of balls of diameter < 1 needed to cover an arbitrary set of diameter 1 in $\mathbb{E}^n$.

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Any ball of diameter $\sqrt{3}$ intersects $\mathbb{S}^{n-1}$ by a cap of radius $< \varphi$, so by (b) of the probabilistic lemma we need at least $\left(\frac{1+o(1)}{\sin\varphi}\right)^n$ such caps to cover X.

Concluding remarks

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