On equicontinuity of families of mappings with one normalization condition by the prime ends

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$$D \ni x = (x_1, \ldots, x_n) \mapsto f(x) = (f_1(x), \ldots, f_n(x)) \in f(D) \qquad (1)$$

Mappings $f : D \to \mathbb{R}^n$

$D \subset \mathbb{R}^n$ is a **domain** $\Leftrightarrow$ D is **connected** and **open** set in $\mathbb{R}^n$

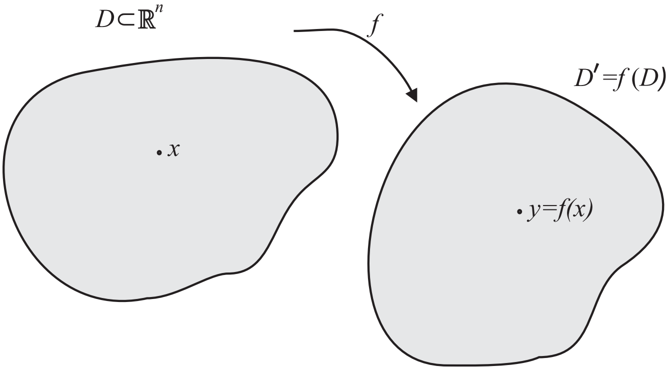

Figure 1: A mapping of a domain D

$\gamma : [a,b] \to \mathbb{R}^n$, $n \geqslant 2$, **length** is the supremum of

$$\sum_{i=1}^{k} |\gamma(t_i) - \gamma(t_{i-1})|$$

over all partitions $a = t_0 \leqslant t_1 \leqslant \ldots \leqslant t_k = b$ of the interval $[a,b]$.

$\rho : \mathbb{R}^n \to [0,\infty]$,

$$\boxed{\rho \in \operatorname{adm}\Gamma \Leftrightarrow \int\limits_{\gamma} \rho\, ds \ \geqslant \ 1} \qquad (2)$$

for all (locally rectifiable) $\gamma \in \Gamma$.

$p \geqslant 1$, the p-**modulus** of Γ is

$$M_p(\Gamma) = \inf_{\rho \in \mathrm{adm}\,\Gamma} \int\limits_{\mathbb{R}^n} \rho^{\,p}(x)\, dm(x)\,. \qquad (3)$$

$$M(\Gamma) := M_n(\Gamma)$$

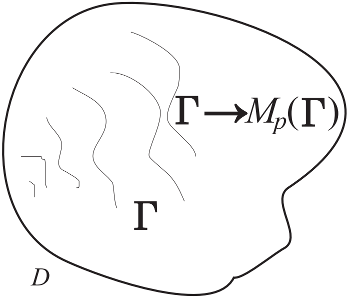

Figure 2: The modulus as real function of family of paths

$\Gamma(E,F,D)$: the family of all continuous curves $\gamma : [0,1] \to \mathbb{R}^n$ such that $\gamma(0) \in E$, $\gamma(1) \in F$, and $\gamma(t) \in D$ for all $t \in (0,1)$.

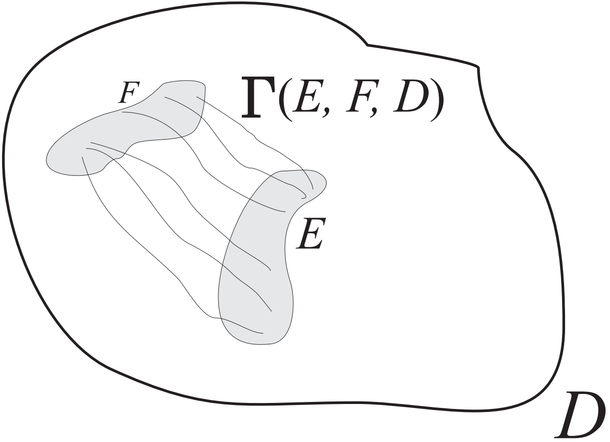

Figure 3: Families of paths joining two fixed sets

Let $x_0 \in \overline{D}$, $x_0 \neq \infty$,

$$B(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| < r\}\,, \quad \mathbb{B}^n = B(0,1)\,, \tag{4}$$

$$S(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| = r\}\,, S_i = S(x_0, r_i)\,, \quad i = 1, 2\,,$$

$$A = A(x_0, r_1, r_2) = \{x \in \mathbb{R}^n : r_1 < |x - x_0| < r_2\}\,.$$

Let $f : D \to \mathbb{R}^n$, $n \geqslant 2$, and let $Q : \mathbb{R}^n \to [0, \infty]$ be a Lebesgue measurable function such that $Q(y) \equiv 0$ for $y \in \mathbb{R}^n \setminus f(D)$. Let $A = A(y_0, r_1, r_2)$. Let $\Gamma_f(y_0, r_1, r_2)$ denotes the family of all paths $\gamma : [a, b] \to D$ such that $f(\gamma) \in \Gamma(S(y_0, r_1), S(y_0, r_2), A(y_0, r_1, r_2))$, i.e., $f(\gamma(a)) \in S(y_0, r_1)$, $f(\gamma(b)) \in S(y_0, r_2)$, and $f(\gamma(t)) \in A(y_0, r_1, r_2)$ for any $a < t < b$. We say that f **satisfies the inverse Poletsky inequality** at $y_0 \in f(D)$ if the relation

$$M(\Gamma_f(y_0, r_1, r_2)) \leqslant \int\limits_A Q(y) \cdot \eta^n(|y - y_0|)\, dm(y) \tag{5}$$

holds for any $0 < r_1 < r_2 < r_0 := \sup\limits_{y \in D'} |y - y_0|$ and any Lebesgue measurable function $\eta : (r_1, r_2) \to [0, \infty]$ such that

$$\int\limits_{r_1}^{r_2} \eta(r)\, dr \geqslant 1\,. \tag{6}$$

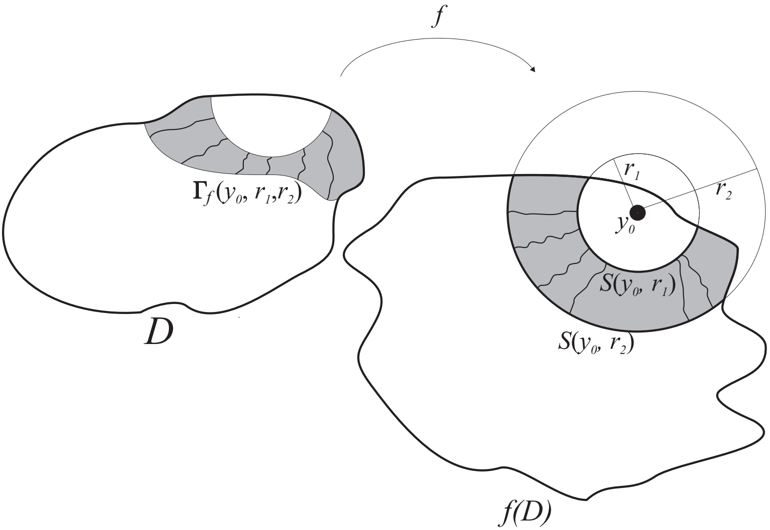

Figure 4: Illustration of the action of a mapping with the inverse Poletsky inequality

- If f is a quairegular, then f satisfies (5) with $Q = K\cdot N(f, D)$, where

$$N(f,D)\ =\ \sup_{y\in\mathbb{R}^n} N(y,f,D)\,, N(y,f,D)\ =\ \mathrm{card}\,\{x\in D: f(x)=y\}\ ,$$

and $K\geqslant 1$ is defined as

$$K=\operatorname{ess\,sup} K_O(x,f)\,, \tag{7}$$

$$K_O(x,f)=\Vert f\,'(x)\Vert^n/J(x,f) \tag{8}$$

for $J(x,f)\neq 0$; $K_O(x,f)=1$ for $f\,'(x)=0$, and $K_O(x,f)=\infty$ for $f\,'(x)\neq 0$, but $J(x,f)=0$ [1]

- If $f\in W_{\rm loc}^{1,n}(D)$ with $K_O(x,f)\in L_{\rm loc}^{n-1}(D)$, then f satisfies (5) with $Q:=K_I(y,f^{\,-1}):=\sum\limits_{x\in f^{\,-1}(y)} K_O(x,f)$ [2]

[1] see, e.g. Theorem 3.2 in [Martio, O., S. Rickman, and J. Väisälä: Definitions for quasiregular mappings. - Ann. Acad. Sci. Fenn. Ser. A1 448, 1969, 1–40] or Theorem 6.7.II in [Rickman, S.: Quasiregular mappings. – Springer-Verlag, Berlin etc., 1993]

[2] see e.g. [Martio, O., V. Ryazanov, U. Srebro, and E. Yakubov: Moduli in modern mapping theory. - Springer Science + Business Media, LLC, New York, 2009]

ω – open set in $\mathbb{R}^k$, $k = 1, \ldots, n-1$.

k-dimensional surface: $\sigma : \omega \to \mathbb{R}^n$

Surface: $k = n - 1$

Jordan surface: $\sigma : \omega \to D$, $\sigma(z_1) \neq \sigma(z_2)$ for $z_1 \neq z_2$.

$$\sigma \sim \sigma(\omega) \subset \mathbb{R}^n \,, \qquad \overline{\sigma} \sim \overline{\sigma(\omega)} \,, \qquad \partial\sigma \sim \overline{\sigma(\omega)} \setminus \sigma(\omega) \tag{9}$$

Cut: σ is Jordan surface which splits D,

$$\boxed{\partial\sigma \cap D = \varnothing \,, \qquad \partial\sigma \cap \partial D \neq \varnothing} \,. \tag{10}$$

Chain: $\sigma_1, \sigma_2, \ldots, \sigma_m, \ldots$

(i) $\overline{\sigma_i} \cap \overline{\sigma_j} = \varnothing$ for all $i \neq j$, $i, j = 1, 2, \ldots$;

(ii) σ_{m-1} and σ_{m+1} are contained in different components of $D \setminus \sigma_m$;

(iii) $\cap\, d_m = \varnothing$, where d_m is a component of $D \setminus \sigma_m$, containing σ_{m+1}.

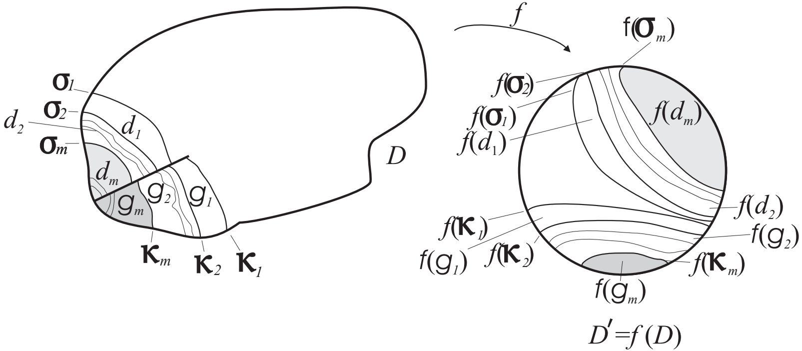

Figure 5: Chains and correspondence between them under some mapping

$\{\sigma_m\}$ and $\{\sigma'_k\}$ are **equivalent** $\Leftrightarrow$

1) d_m contains all d'_k except finitely many, $\forall\ m = 1, 2, \ldots$,

2) d'_k contains all d'_k except finitely many, $\forall\ k = 1, 2, \ldots$,

End $\equiv$ equivalence class of chains of cuts of D. End K is a **prime end** $\equiv K$ contains $\{\sigma_m\}$: for some continuum $C \subset D$

$$\boxed{\lim_{m\to\infty} M(\Gamma(C, \sigma_m, D)) = 0} \tag{11}$$

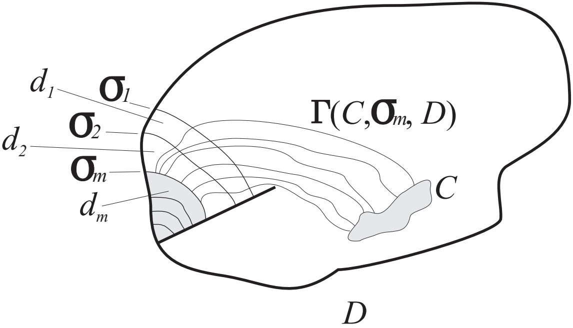

Figure 6: A prime end in some domain

Locally quasiconformal boundary

∂D is **locally quasiconformal** $\Leftrightarrow \forall\ x_0 \in \partial D\ \exists\ U \supset \{x_0\}$ and a quasiconformal mapping φ of U onto $\mathbb{B}^n$: $\varphi(\partial D \cap U) = \mathbb{B}^n_+$

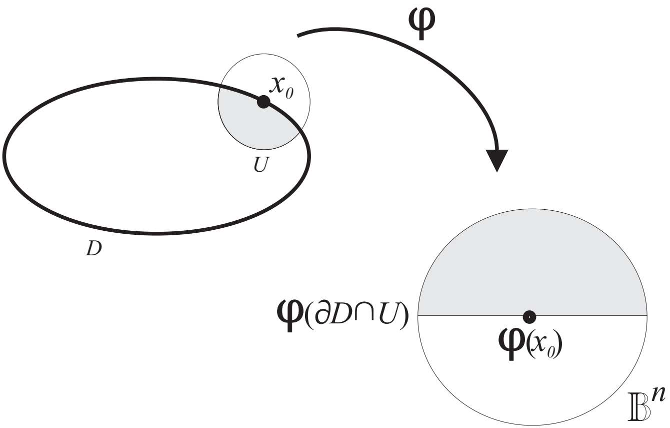

Figure 7: A domain with a locally quasiconformal boundary

D is **regular** $\Leftrightarrow$ D is bounded and D can be mapped quasiconformally onto a domain with a locally quasiconformal boundary

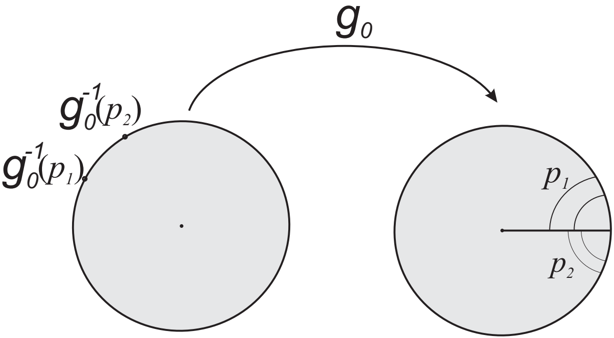

Figure 8: An example of a regular domain

$\overline{D}_P$ is the completion of a regular domain D by its prime ends, g_0 is a quasiconformal mapping of D_0 with locally quasiconformal boundary onto D

$$\boxed{\rho_0(p_1, p_2) = \left|\widetilde{g_0}^{\,-1}(p_1) - \widetilde{g_0}^{\,-1}(p_2)\right|} \qquad (12)$$

$\widetilde{g_0}$ is the extension of g_0 onto $\overline{D_0}$

We say that ∂D is **weakly flat at a point** $x_0 \in \partial D$ if, for every neighborhood U of the point x_0 and every number $P > 0$, there is a neighborhood $V \subset U$ of x_0 such that

$$M(\Gamma(E, F, D)) \geqslant P \tag{13}$$

for all continua E and F in D intersecting ∂U and ∂V.

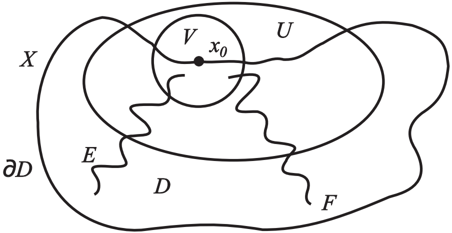

Figure 9: A domain with weakly flat boundary

Given domains $D, D' \subset \mathbb{R}^n$, $n \geqslant 2$, points $a \in D$, $b \in D'$ and a Lebesgue measurable function $Q : D' \to [0, \infty]$ denote $\mathfrak{S}_{a,b,Q}(D, D')$ a family of all open discrete and closed mappings f of D onto D', satisfying the relation (5) for any $y_0 \in D'$, while $f(a) = b$. The following statement holds.

Theorem 1

Assume that, D has a weakly flat boundary, any component of which does not degenerate into a point. If $Q \in L^1(D')$ and D' is regular, then any $f \in \mathfrak{S}_{a,b,Q}(D, D')$ has a continuous extension $\overline{f} : \overline{D} \to \overline{D'}_P$, $\overline{f}(\overline{D}) = \overline{D'}_P$, and, in addition, a family $\mathfrak{S}_{a,b,Q}(\overline{D}, \overline{D'})$ which consists of all extended mappings $\overline{f} : \overline{D} \to \overline{D'}_P$, is equicontinuous in $\overline{D}$.

The results of the report are published in Ukrainian Mathematical Journal:

Ilkevych N.S., Sevost'yanov E.A. On equicontinuity of the families of mappings with one normalization condition in terms of prime ends. *Ukrainian Mathematical Journal*, 74 (6): 936–945, 2022.

Figure 10: Ukrainian Mathematical Journal

It is over ! Thank you very much !