Poincaré-Reeb graphs of real algebraic domains

Miruna-Ştefana Sorea

SISSA, Trieste, Italy and ULBS, Sibiu, Romania

Algebraic and Geometric Methods of Analysis (AGMA 2023)

May 30, 2023
Setting and goals

- **objects:** closed topological subsurfaces of a real affine plane whose boundary consists of disjoint smooth connected components of real algebraic plane curves: algebraic domains $\mathcal{D}$. 

![Diagram of algebraic domains and curves]
Algebraic domains

Example

\( \overline{C_1} \) of equation \((y^2 - (x - 1)(x - 2)(x - 3) = 0)\)
\( \overline{C_2} \) of equation \((y^2 - x(x - 4)(x - 5) = 0)\)

A ring surface:
Setting and goals

- **goal:** measure the **non-convexity** of algebraic domains $D$. 
• **goal:** measure the **non-convexity** of algebraic domains $\mathcal{D}$.
  
  $\rightarrow$ in a certain direction of projection.
How to measure non-convexity?

**Idea:** collapse all vertical segments contained in the algebraic domain
How to measure non-convexity?

Idea: collapse all vertical segments contained in the algebraic domain

⇒ construct the Poincaré-Reeb graph associated to an algebraic domain and to a direction of projection.
Overview

• First part: **strict local minima**

![Diagram of strict local minima]

• Second part: **a purely topological description** in which our construction of Poincaré-Reeb graphs can be applied

• Third part: **real algebraic domains**

![Diagram of real algebraic domains]
References:


Table of Contents

1 Part I-Effective construction in the case of strict local minima

2 Part II-Poincaré-Reeb graphs of domains of (weakly) finite type

3 Part III-Realization of Poincaré-Reeb graphs by algebraic domains
• **objects:** polynomial functions \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \), \( f(0, 0) = 0 \) such that \( O \) is a strict local minimum;

• **goal:** study the **real** Milnor fibres of the polynomial (i.e. the level curves \( f(x, y) = \varepsilon \)), for \( 0 < \varepsilon \ll 1 \), in a small enough neighbourhood of the origin).

\[
f(x, y) = x^2 + y^2
\]
Whenever the origin is a **Morse** strict local minimum the **small enough** level curves are boundaries of **convex** topological disks.
Question (Giroux asked Popescu-Pampu, 2004)

Are the small enough level curves of $f$ near strict local minima always boundaries of **convex** disks?

**Counterexample** by M. Coste: $f(x, y) = x^2 + (y^2 - x)^2$. 
• Problem: understand these non-convexity phenomena.
• Subproblem 1: construct non-Morse strict local minima whose nearby small levels are far from being convex.
Question

What *combinatorial object* can encode the shape by measuring the non-convexity of a smooth and compact connected component of an algebraic curve in $\mathbb{R}^2$?
Definition

Two points of $\mathcal{D}$ are equivalent if they belong to the same connected component of a fibre of the projection $\Pi : \mathbb{R}^2 \to \mathbb{R}$, $\Pi(x, y) := x$. 

The Poincaré-Reeb graph

associated to a curve and to a direction $x$
Theorem ([Sor22a])

The Poincaré-Reeb graph is a transversal tree: it is a plane tree whose open edges are transverse to the foliation induced by the function $x$; its vertices are endowed with a total preorder relation induced by the function $x$. 
The \textbf{asymptotic} Poincaré-Reeb tree

- small enough level curves;
- near a strict local minimum.

\textbf{Theorem ([Sor22a])}

The asymptotic Poincaré-Reeb tree \textit{stabilises}. It is a \textit{rooted} tree; the total preorder relation on its vertices is \textit{strictly monotone} on each geodesic starting from the root.
• **Characterise** all possible topological types of asymptotic Poincaré-Reeb trees.

• **Construct** a family of polynomials realising a large class of transversal trees as their Poincaré-Reeb trees.
Main result - Part I

- introduction of new combinatorial objects;
- polar curve, discriminant curve;
- genericity hypotheses ($x > 0$);
- univariate case: explicit construction of separable snakes;
- a result of realisation of a large class of Poincaré-Reeb trees.

Theorem ([Sor18])

Given any separable positive generic rooted transversal tree, we construct the equation of a real bivariate polynomial with isolated minimum at the origin which realises the given tree as a Poincaré-Reeb tree.
Main result - Part I

- introduction of new combinatorial objects;
- polar curve, discriminant curve;
- genericity hypotheses \((x > 0)\);
- univariate case: explicit construction of separable snakes;
- a result of realisation of a large class of Poincaré-Reeb trees.

**Theorem ([Sor18])**

*Given any separable positive generic rooted transversal tree, we construct the equation of a real bivariate polynomial with isolated minimum at the origin which realises the given tree as a Poincaré-Reeb tree.*
Tool 1: The polar curve

\[ \Gamma(f, x) := \left\{ (x, y) \in \mathbb{R}^2 \mid \frac{\partial f}{\partial y}(x, y) = 0 \right\} \]

It is the set of points where the tangent to a level curve is vertical.
Tool 2: Choosing a generic projection

Avoid vertical inflections:

Avoid vertical bitangents:
The generic asymptotic Poincaré-Reeb tree

**Theorem ([Sor22b])**

*In the asymptotic case, if the direction $x$ is generic, then we have a **total order** relation and a **complete binary** tree.*

Two inequivalent trees
Tool 3: The discriminant locus

\[ \Phi : \mathbb{R}^2_{x,y} \to \mathbb{R}^2_{x,z}, \Phi(x, y) = \left( x, f(x, y) \right). \]

The critical locus of \( \Phi \) is the polar curve \( \Gamma(f, x) \).

The discriminant locus of \( \Phi \) is the critical image \( \Delta = \Phi(\Gamma) \).
Genericity hypotheses

The family of polynomials that we construct satisfies the following two genericity hypotheses:

- the curve $\Gamma_+$ is reduced;
- the map $\Phi|_{\Gamma_+} : \Gamma_+ \to \Delta_+$ is a homeomorphism.
1. Positive asymptotic snake

To any positive (i.e. for $x > 0$) generic asymptotic Poincaré-Reeb tree we can associate a permutation $\sigma$, called the positive asymptotic snake.
2. Arnold’s snake (one variable)

One can associate a permutation to a Morse polynomial, by considering two total order relations on the set of its critical points: **Arnold’s snake**.

\[
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 \\
4 & 5 & 1 & 3 & 2
\end{pmatrix}
\]
2. Arnold’s snake (one variable)

The study of asymptotic forms of the graphs of one variate polynomials \( f(x_0, y) \), for \( x_0 \) tending to zero.

**Theorem ([Sor18])**

\[ \sigma = \tau. \]
Idea of the proof

The interplay between the polar curve and the discriminant curve:

\[ \sigma = \tau = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \]
The construction

Subquestion

Given a generic rooted transversal tree, can we construct the equation of a real bivariate polynomial with isolated minimum at the origin which realises the given tree as a Poincaré-Reeb tree?
The construction

Subquestion

Given a generic rooted transversal tree, can we construct the equation of a real bivariate polynomial with isolated minimum at the origin which realises the given tree as a Poincaré-Reeb tree?

Theorem ([Sor18])

We give a positive constructive answer: we construct a family of polynomials that realise all separable positive generic rooted transversal trees.
\[ \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 5 & 3 & 4 & 2 \end{pmatrix} \]
\[
\sigma = \begin{pmatrix}
1 & 2 & 3 & 4 & 5 \\
1 & 5 & 3 & 4 & 2
\end{pmatrix}
\]
$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 5 & 3 & 4 & 2 \end{pmatrix}$
\[ \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 5 & 3 & 4 & 2 \end{pmatrix} \]
$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 5 & 3 & 4 & 2 \end{pmatrix}$
\[ \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 5 & 3 & 4 & 2 \end{pmatrix} \]
Separable permutations

\[ \sigma = (1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7) = ((\Box \boxplus \Box) \boxdot (\Box \boxplus \Box)) \boxdot (\Box \boxplus (\Box \boxminus \Box)). \]
Nonseparable permutation - example

\[
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 \\
1 & 4 & 2 & 5 & 3
\end{pmatrix}
\]
Definition

A positive generic rooted transversal tree is separable if its associated permutation is separable.
Question

Given a separable snake $\sigma$, is it possible to construct a Morse polynomial $Q : \mathbb{R} \to \mathbb{R}$ that realises $\sigma$?
Example

\[
= \left( \begin{array}{c}
\square \oplus (\square \ominus \square) \\
\end{array} \right) \oplus \left( \begin{array}{c}
\square \ominus \square \\
\end{array} \right) = \left( \begin{array}{ccccc}
1 & 2 & 3 & 4 & 5 \\
1 & 3 & 2 & 5 & 4 \\
\end{array} \right).
\]
The contact tree

\[
\begin{align*}
a_1(x) &= 0, \\
a_2(x) &= x^2, \\
a_3(x) &= x^2 + x^3, \\
a_4(x) &= x^1, \\
a_5(x) &= x^1 + x^2.
\end{align*}
\]
Answer in the univariate case

Theorem ([Sor20])

Consider \( m \in \mathbb{N} \) and fix a separable \((m + 1)\)-snake \( \sigma : \{1, 2, \ldots, m + 1\} \to \{1, 2, \ldots, m + 1\} \) such that \( \sigma(m) > \sigma(m + 1) \). Construct the polynomials \( a_i(x) \in \mathbb{R}[x] \) such that their contact tree is one of the binary separating trees of \( \sigma \). Let \( Q_x(y) \in \mathbb{R}[x][y] \) be

\[
Q_x(y) := \int_0^y \prod_{i=1}^{m+1} (t - a_i(x)) \, dt.
\]

Then \( Q_x(y) \) is a one variable Morse polynomial and for sufficiently small \( x > 0 \), the Arnold snake associated to \( Q_x(y) \) is \( \sigma \).
Construction of the desired bivariate polynomial $f$

Theorem ([Sor18])

Let $\sigma$ be a separable $(m + 1)$-snake, with $m$ an even integer, $\sigma(m) > \sigma(m + 1)$. Let $f \in \mathbb{R}[x, y]$ be constructed as follows:

(a) construct $Q_x(y) \in \mathbb{R}[x][y]$, 

$$Q_x(y) := \int_0^y \prod_{i=1}^{m+1} (t - a_i(x)) \, dt,$$

by choosing the polynomials $a_i(x) \in \mathbb{R}[x]$ such that their contact tree is one of the binary separating trees of $\sigma$.

(b) take $f(x, y) := x^2 + Q_x(y)$.

Then $f$ has a strict local minimum at the origin and the positive asymptotic snake of $f$ is the given $\sigma$. 

Miruna-Ştefana Sorea (SISSA)
Properties of $f$

$$f(x, y) := x^2 + \int_0^y \prod_{i=1}^{m+1} (t - a_i(x)) \, dt.$$ 

- Its positive generic asymptotic Poincaré-Reeb tree:

- It has a strict local minimum at the origin:
Pairwise distinct polynomials $a_i(x) \in \mathbb{R}[x]$ that pass through a common zero at the origin.
Positive-negative contact trees (one variable)\(^3\)

\[ x + x^4 \]
\[ x^3 + x^4 \]
\[ x^2 + x^4 \]
\[ x^4 + x^5 \]

\(^2\)Picture from [Ghy17]
\(^3\)É. Ghys - A singular mathematical promenade, 2017
Positive-negative contact trees (one variable)

$y = x + x^4$

$y = x^2 + x^4$

$y = x^3 + x^4$

$y = x^4 + x^5$

---

$^2$Picture from [Ghy17]

$^3$É. Ghys - A singular mathematical promenade, 2017
Flip-Flop Algorithm

? \[ \tilde{O} \]

a. 

b. 

c. 

d. 

a. 

b. 

c. 

d.
Table of Contents

1. Part I-Effective construction in the case of strict local minima

2. Part II-Poincaré-Reeb graphs of domains of (weakly) finite type

3. Part III-Realization of Poincaré-Reeb graphs by algebraic domains
General context
Vertical planes

→ motivation: after the collapsing procedure, we end up with a new vertical plane
Vertical planes

→ motivation: after the collapsing procedure, we end up with a new vertical plane

→ a pair \((\mathcal{P}, \pi)\) such that \(\mathcal{P}\) is a topological space homeomorphic to \(\mathbb{R}^2\), endowed with an orientation, and \(\pi : \mathcal{P} \rightarrow \mathbb{R}\) is a locally trivial topological fibration
Vertical planes

→ motivation: after the collapsing procedure, we end up with a new vertical plane

→ a pair \((\mathcal{P}, \pi)\) such that \(\mathcal{P}\) is a topological space homeomorphic to \(\mathbb{R}^2\), endowed with an orientation, and \(\pi : \mathcal{P} \to \mathbb{R}\) is a locally trivial topological fibration

→ the **canonical affine vertical plane** is \((\mathbb{R}^2, x : \mathbb{R}^2 \to \mathbb{R})\)
The topological critical set

\[ \Sigma_{\text{top}}(C) : \text{points } p \in C \text{ in whose neighborhoods the restriction } \pi|_C \text{ is not a local homeomorphism onto its image.} \]
Domain of finite type

\((\mathcal{P}, \pi)\) is a vertical plane
\(\mathcal{D} \subset \mathcal{P}\) is a closed subset homeomorphic to a surface with non-empty boundary \(\mathcal{C}\)

Definition

We say that \(\mathcal{D}\) is a **domain of finite type** in \((\mathcal{P}, \pi)\) if:

1. the restriction \(\pi|_{\mathcal{D}} : \mathcal{D} \to \mathbb{R}\) is proper;
2. the topological critical set \(\Sigma_{\text{top}}(\mathcal{C})\) is finite.

A domain of finite type is **generic** if no two topological critical points of its boundary lie on the same vertical line.
Domains which are not of finite type:
Vertical equivalence

$X$ and $X'$ subsets of the vertical planes $(\mathcal{P}, \pi)$, resp. $(\mathcal{P}', \pi')$.

**Definition**

We say that $X \approx_v X'$, if there exist orientation preserving homeomorphisms $\Phi : \mathcal{P} \to \mathcal{P}'$ and $\psi : \mathbb{R} \to \mathbb{R}$ such that $\Phi(X) = X'$ and the following diagram is commutative:

$$
\begin{array}{ccc}
\mathcal{P} & \xrightarrow{\Phi} & \mathcal{P}' \\
\downarrow{\pi} & & \downarrow{\pi'} \\
\mathbb{R} & \xrightarrow{\psi} & \mathbb{R}
\end{array}
$$
Consider the canonical affine vertical plane \((\mathbb{R}^2, x)\).

The vertical equivalence preserves the horizontal order:

if \(x(P_i) < x(P_j)\) and \(P'_i = \Phi(P_i), P'_j = \Phi(P_j)\),
then \(x(P'_i) < x(P'_j)\).

\[\rightarrow\] important for topological critical points.
Proposition

Let $D$ and $D'$ be compact connected domains of finite type in vertical planes, with Poincaré–Reeb graphs $G$ and $G'$. Assume that both are generic. Then:

$$D \approx_v D' \iff G \approx_v G'.$$
Domains which are not vertically equivalent
Other invariants?

Two generic real algebraic domains homeomorphic to discs with the same permutation \((1 2 3 4 5 6)\), but which are not vertically equivalent.
Proposition

Let $\mathcal{D}$ be a compact domain of finite type in a vertical plane. Then $\mathcal{D}$ and its Poincaré–Reeb graph have the same number of connected components and the same Euler characteristic.

Idea of the proof: integration with respect to the Euler characteristic.
Proposition

If $D \subset (P, \pi)$ is (homeomorphic to) a disk, then the Poincaré–Reeb graph of $D$ is a tree.
Table of Contents

1 Part I-Effective construction in the case of strict local minima

2 Part II-Poincaré-Reeb graphs of domains of (weakly) finite type

3 Part III-Realization of Poincaré-Reeb graphs by algebraic domains
Main result

Theorem (Bodin, Popescu-Pampu, Sorea, 2022)

Any compact connected generic transversal graph can be realized as a Poincaré–Reeb graph of an algebraic domain.\(^a\)

\(^a\)[BPPS23], https://arxiv.org/pdf/2207.06871.pdf
Strategy of the proof

• Step 1: we realize the generic transversal graph $G$ as a Poincaré–Reeb graph of a finite type domain defined by a smooth function;
Strategy of the proof

- Step 1: we realize the generic transversal graph $G$ as a Poincaré–Reeb graph of a finite type domain defined by a smooth function;

- Step 2: we use a Weierstrass-type theorem that approximates any smooth function by a polynomial function;
Strategy of the proof

• Step 1: we realize the generic transversal graph $G$ as a Poincaré–Reeb graph of a finite type domain defined by a smooth function;

• Step 2: we use a Weierstrass-type theorem that approximates any smooth function by a polynomial function;

• Step 3: we adapt this Weierstrass-type theorem in order to control vertical tangents, and we realize $G$ as the Poincaré–Reeb graph of a generic finite type algebraic domain.
Definition

We say that $\mathcal{D}$ is a **domain of weakly finite type** if:

1. the restriction $\pi|_C : C \to \mathbb{R}$ is proper;
2. the topological critical set $\Sigma_{\text{top}}(C)$ is finite.
Domains of weakly finite type

Definition

We say that $D$ is a **domain of weakly finite type** if:

1. the restriction $\pi|_c : C \to \mathbb{R}$ is proper;
2. the topological critical set $\Sigma_{top}(C)$ is finite.

A domain of weakly finite type is called **generic** if no two topological critical points of $C$ lie on the same vertical line.
Non-compact Poincaré–Reeb graphs

When $\mathcal{C}$ is homeomorphic to a line, we distinguish three cases, depending on the position of $\mathcal{D}$ and of the branches of $\mathcal{C}$. 
Theorem (Bodin, Popescu-Pampu, Sorea, 2022)

Let $G$ be a connected, non-compact, generic, transversal tree. Let $G'$ be the compact tree obtained from $G$. If $G'$ can be realized by a connected real algebraic curve, then $G$ can be realized as the Poincaré–Reeb graph of a simply connected, non-compact algebraic domain in $(\mathbb{R}^2, x)$. 
Theorem (Bodin, Popescu-Pampu, Sorea, 2022)

Let $G$ be a connected, non-compact, generic, transversal tree. Let $G'$ be the compact tree obtained from $G$. If $G'$ can be realized by a connected real algebraic curve, then $G$ can be realized as the Poincaré–Reeb graph of a simply connected, non-compact algebraic domain in $(\mathbb{R}^2, x)$. 
Strategy of proof - Case A

\[ G \]

\[ (f \cdot g = \varepsilon) \]

\[ (g = 0) \]

\[ G' \]

\[ (f = 0) \]
Strategy of proof - Case B

\[ \mathcal{G} \]
Case C is the complement of Case A:
Proposition

*The interior graph* $G$ *of a domain* $\mathcal{D}$ *of weakly finite type determines its exterior graph* $G^c$. 

*Miruna-Ştefana Sorea (SISSA)*

*Poincaré-Reeb graphs*
Thank you!
Bibliography:


