Poincaré-Reeb graphs of real algebraic domains

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Algebraic and Geometric Methods of Analysis (AGMA 2023)

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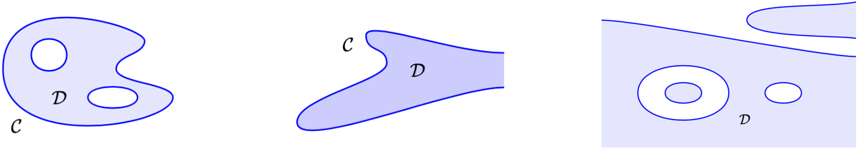
Setting and goals

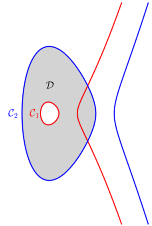
- **objects**: closed topological subsurfaces of a real affine plane whose boundary consists of disjoint smooth connected components of real algebraic plane curves: **algebraic domains** $\mathcal{D}$.



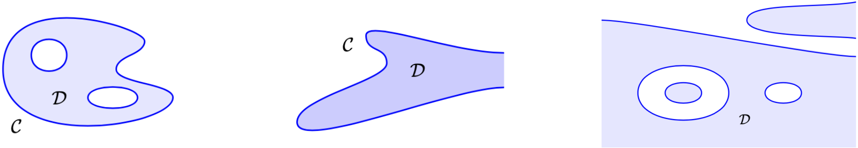
Algebraic domains

Example

$\overline{\mathcal{C}_1}$ *of equation* $(y^2 - (x-1)(x-2)(x-3) = 0)$
$\overline{\mathcal{C}_2}$ *of equation* $(y^2 - x(x-4)(x-5) = 0)$
A ring surface:

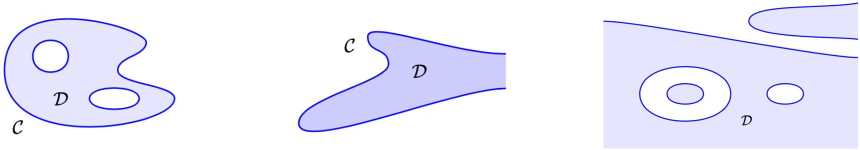


Setting and goals



- **goal:** measure the **non-convexity** of algebraic domains $\mathcal{D}$.

Setting and goals



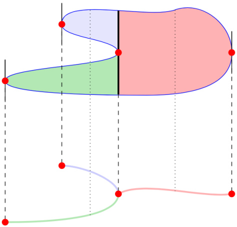
- **goal:** measure the **non-convexity** of algebraic domains $\mathcal{D}$.
  –> in a certain direction of projection.

How to measure non-convexity?

Idea: collapse all vertical segments contained in the algebraic domain

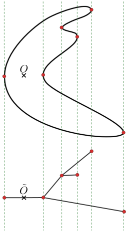
How to measure non-convexity?

Idea: collapse all vertical segments contained in the algebraic domain
$\rightsquigarrow$ construct **the Poincaré-Reeb graph** associated to an algebraic domain and to a direction of projection.

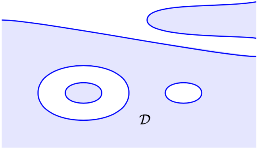


Overview

- First part: **strict local minima**



- Second part: **a purely topological description** in which our construction of Poincaré-Reeb graphs can be applied
- Third part: **real algebraic domains**



References:

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Arnaud Bodin, Patrick Popescu-Pampu, and Miruna-Ştefana Sorea. "Poincaré-Reeb graphs of real algebraic domains". In: *Revista Matemática Complutense* (2023). url: https://link.springer.com/article/10.1007/s13163-023-00469-y (cit. on p. 68).

Étienne Ghys. *A singular mathematical promenade*. ENS Éditions, Lyon, 2017, pp. viii+302. url: http://ghys.perso.math.cnrs.fr/bricabrac/promenade.pdf (cit. on pp. 49, 50).

Miruna-Ştefana Sorea. "The shapes of level curves of real polynomials near strict local minima". PhD thesis. Université de Lille, 2018. url: https://hal.archives-ouvertes.fr/tel-01909028v1 (cit. on pp. 20, 21, 29, 31, 32, 46).

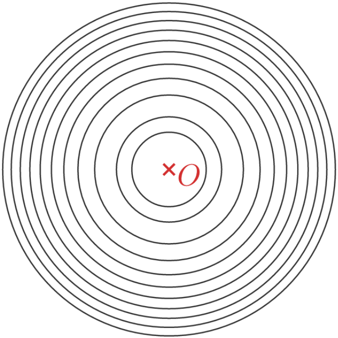
Miruna-Ştefana Sorea. "Constructing Separable Arnold Snakes of Morse Polynomials". In: *Portugaliae Mathematica* (2020). doi: 10.4171/PM/2050 (cit. on p. 45).

Miruna-Ştefana Sorea. "Measuring the local non-convexity of real algebraic curves". In: *Journal of Symbolic Computation* (2022). doi: 10.1016/j.jsc.2020.07.017 (cit. on pp. 17, 18).

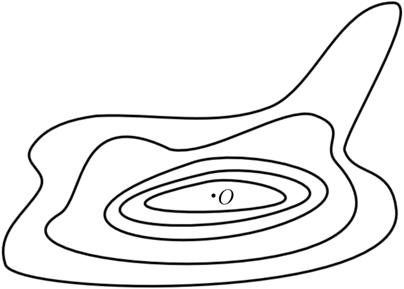
Miruna-Ştefana Sorea. "Permutations encoding the local shape of level curves of real polynomials via generic projections". In: *Annales de l'Institut Fourier* (2022). doi: 10.5802/aif.3479 (cit. on p. 24).

Table of Contents

- **objects:** polynomial functions $f : \mathbb{R}^2 \to \mathbb{R}$, $f(0,0) = 0$ such that O is a strict local minimum;
- **goal:** study the real Milnor fibres of the polynomial (i.e. the level curves $(f(x,y) = \varepsilon)$, for $0 < \varepsilon \ll 1$, in a small enough neighbourhood of the origin).



$f(x,y) = x^2 + y^2$

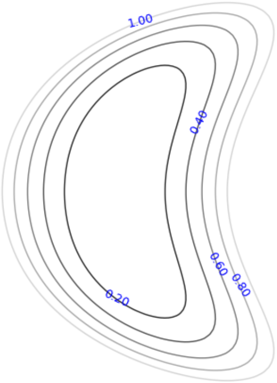


Whenever the origin is a Morse strict local minimum the **small enough** level curves are boundaries of convex topological disks.

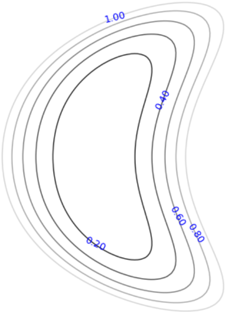
Question (Giroux asked Popescu-Pampu, 2004)

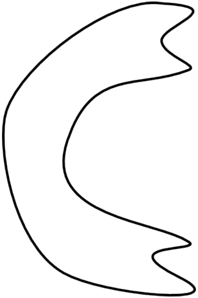
*Are the small enough level curves of f near strict local minima always boundaries of **convex** disks?*

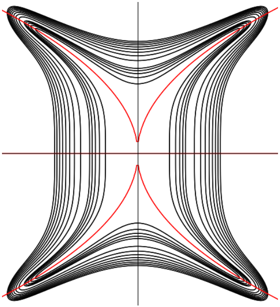
Counterexample by M. Coste: $f(x, y) = x^2 + (y^2 - x)^2$.



- Problem: understand these **non-convexity** phenomena.
- Subproblem 1: **construct non-Morse strict local minima** whose nearby small levels are far from being convex.

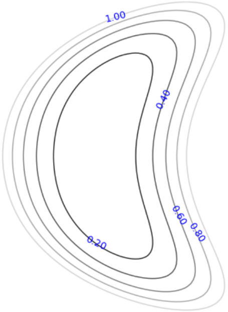


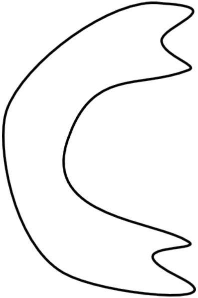


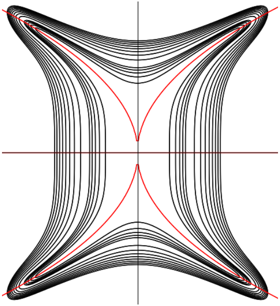


Question

What ***combinatorial object*** *can encode the shape by* ***measuring the non-convexity*** *of a smooth and compact connected component of an algebraic curve in* $\mathbb{R}^2$*?*





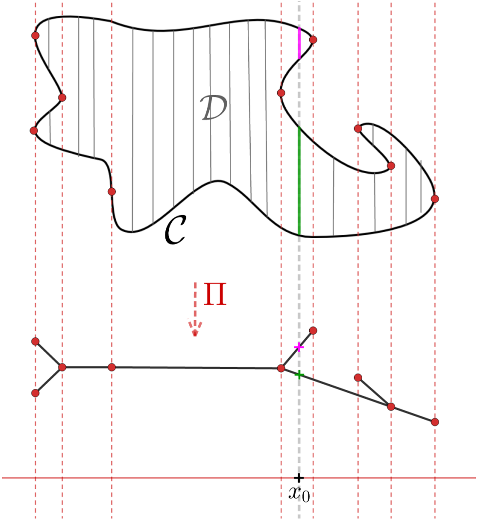


The Poincaré-Reeb graph

associated to a curve and to a direction x

Definition

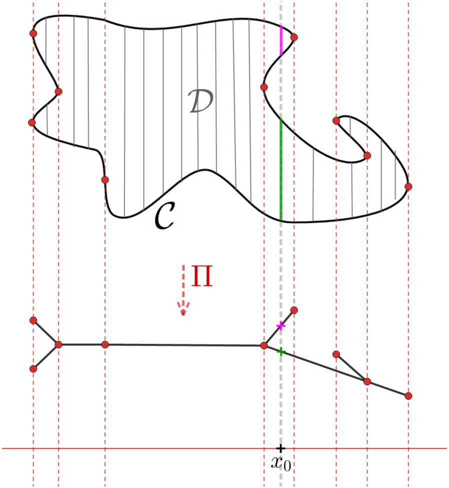
Two points of $\mathcal{D}$ are equivalent if they belong to the same connected component of a fibre of the projection $\Pi : \mathbb{R}^2 \to \mathbb{R}$, $\Pi(x, y) := x$.



The Poincaré-Reeb tree

Theorem ([Sor22a])

*The Poincaré-Reeb graph is a **transversal tree**: it is a **plane tree** whose open edges are **transverse to the foliation** induced by the function x; its vertices are endowed with a **total preorder** relation induced by the function x.*



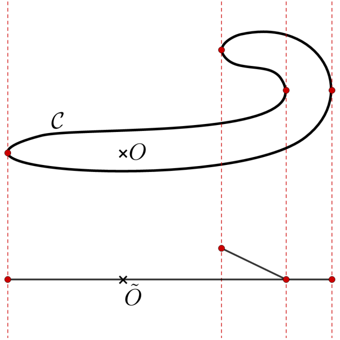
The **asymptotic** Poincaré-Reeb tree

-small enough level curves;
-near a strict local minimum.

Theorem ([Sor22a])

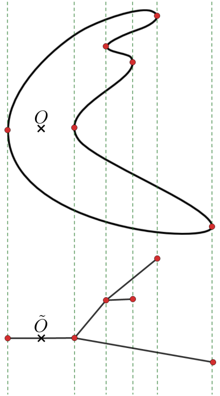
The asymptotic Poincaré-Reeb tree ***stabilises****. It is a* ***rooted*** *tree; the total preorder relation on its vertices is* ***strictly monotone*** *on each geodesic starting from the root.*

Impossible asymptotic configuration:



Reformulation

- Characterise all possible topological types of asymptotic Poincaré-Reeb trees.
- Construct a family of polynomials realising a large class of transversal trees as their Poincaré-Reeb trees.



Main result - Part I

- introduction of new combinatorial objects;
- polar curve, discriminant curve;
- genericity hypotheses ($x > 0$);
- univariate case: explicit construction of separable snakes;
- a result of realisation of a large class of Poincaré-Reeb trees.

Main result - Part I

- introduction of new combinatorial objects;
- polar curve, discriminant curve;
- genericity hypotheses ($x > 0$);
- univariate case: explicit construction of separable snakes;
- a result of realisation of a large class of Poincaré-Reeb trees.

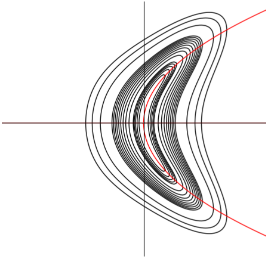
Theorem ([Sor18])

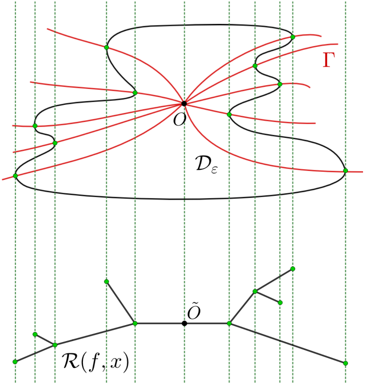
*Given any **separable positive generic rooted transversal tree**, we construct the equation of a real bivariate polynomial with isolated minimum at the origin which realises the given tree as a Poincaré-Reeb tree.*

Tool 1 : The polar curve

$$\Gamma(f, x) := \left\{ (x, y) \in \mathbb{R}^2 \;\middle|\; \frac{\partial f}{\partial y}(x, y) = 0 \right\}$$

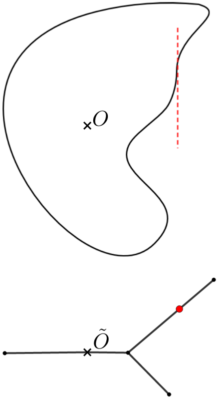
It is the set of points where the tangent to a level curve is vertical.



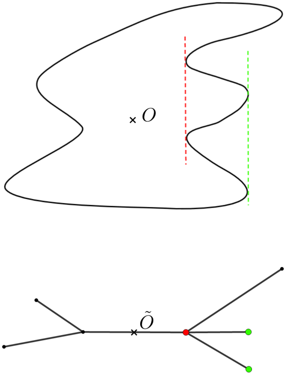


Tool 2 : Choosing a generic projection

Avoid vertical inflections:



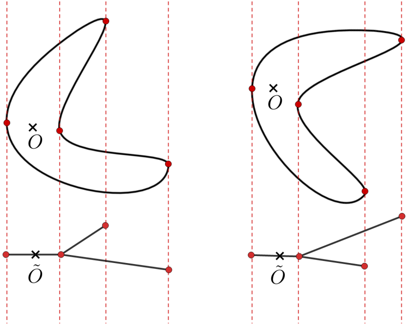
Avoid vertical bitangents:



The generic asymptotic Poincaré-Reeb tree

Theorem ([Sor22b])

*In the asymptotic case, if the direction x is generic, then we have a **total order** relation and a **complete binary** tree.*



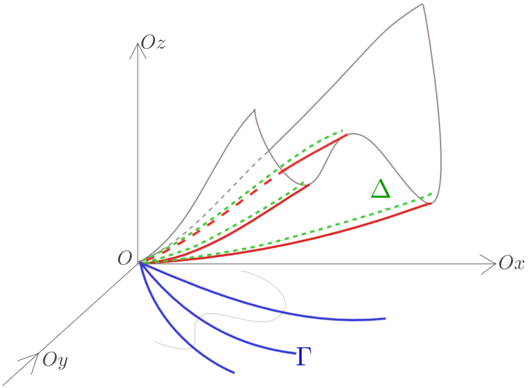
Two inequivalent trees

Tool 3: The discriminant locus

$$\Phi : \mathbb{R}^2_{x,y} \to \mathbb{R}^2_{x,z}, \Phi(x,y) = \Big(x, f(x,y)\Big).$$

The critical locus of Φ is the polar curve $\Gamma(f,x)$.

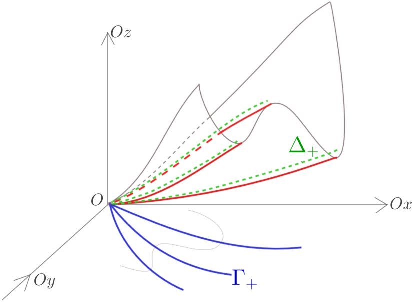
The discriminant locus of Φ is the critical image $\Delta = \Phi(\Gamma)$.



Genericity hypotheses

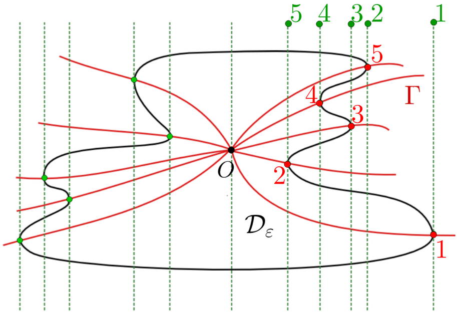
The family of polynomials that we construct satisfies the following two genericity hypotheses:

- the curve Γ_+ is **reduced**;
- the map $\Phi_{|\Gamma_+} : \Gamma_+ \to \Delta_+$ is a **homeomorphism**.



1. Positive asymptotic snake

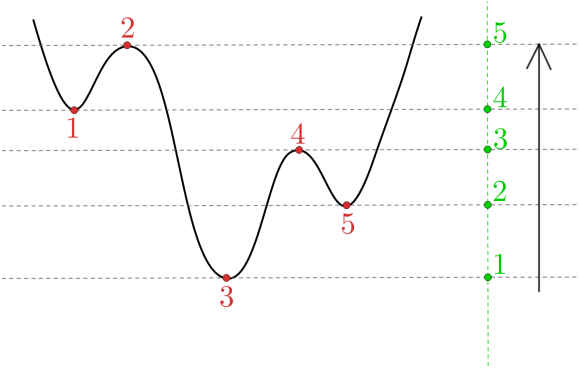
To any positive (i.e. for $x > 0$) generic asymptotic Poincaré-Reeb tree we can associate a permutation σ, called the **positive asymptotic snake**.



$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 5 & 3 & 4 & 2 \end{pmatrix}$$

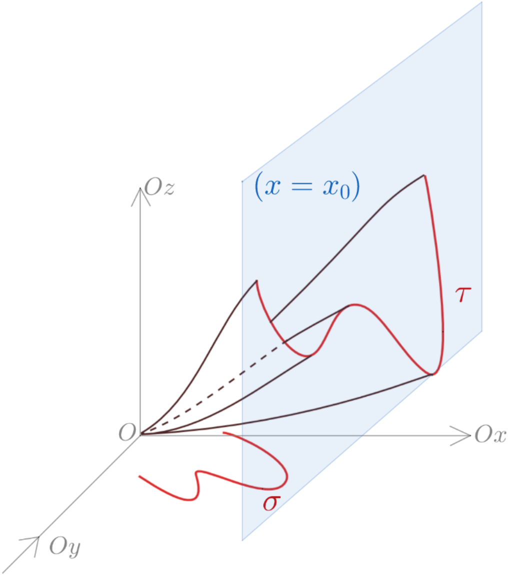
2. Arnold's snake (one variable)

One can associate a permutation to a **Morse polynomial**, by considering two total order relations on the set of its critical points: **Arnold's snake**.



$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 1 & 3 & 2 \end{pmatrix}$$

2. Arnold's snake (one variable)



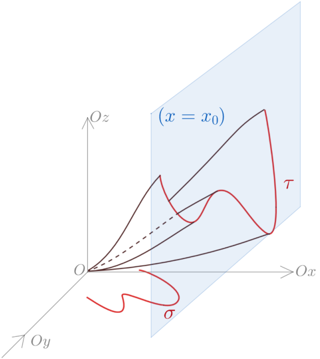
The study of asymptotic forms of **the graphs of one variate polynomials** $f(x_0, y)$, for x_0 tending to zero.

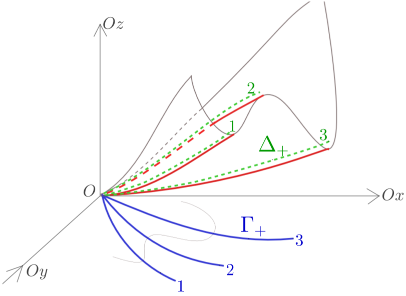
Theorem ([Sor18])

$$\sigma = \tau.$$

Idea of the proof

The interplay between the polar curve and the discriminant curve:





$$\sigma = \tau = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

The construction

Subquestion

Given a generic rooted transversal tree, can we construct the equation of a real bivariate polynomial with isolated minimum at the origin which realises the given tree as a Poincaré-Reeb tree?

The construction

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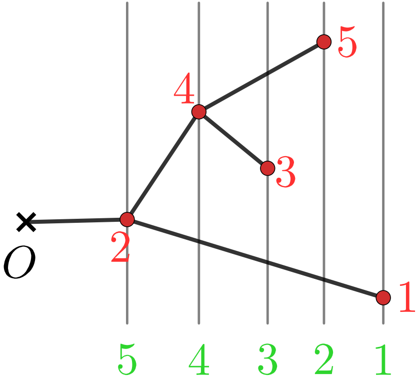
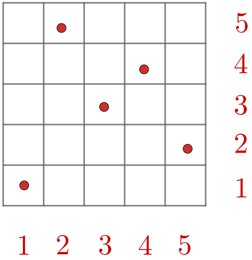
Given a generic rooted transversal tree, can we construct the equation of a real bivariate polynomial with isolated minimum at the origin which realises the given tree as a Poincaré-Reeb tree?

Theorem ([Sor18])

We give a ***positive constructive answer****: we construct a family of polynomials that realise all* ***separable*** *positive generic rooted transversal trees.*



$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 5 & 3 & 4 & 2 \end{pmatrix}$$



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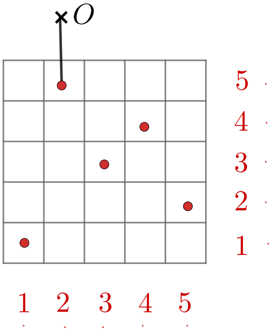

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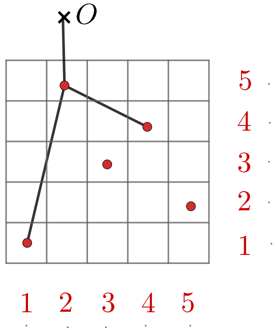
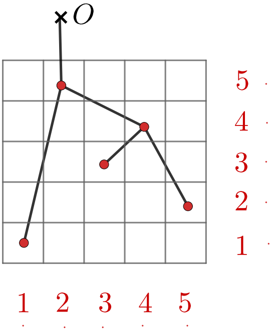


$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 5 & 3 & 4 & 2 \end{pmatrix}$$



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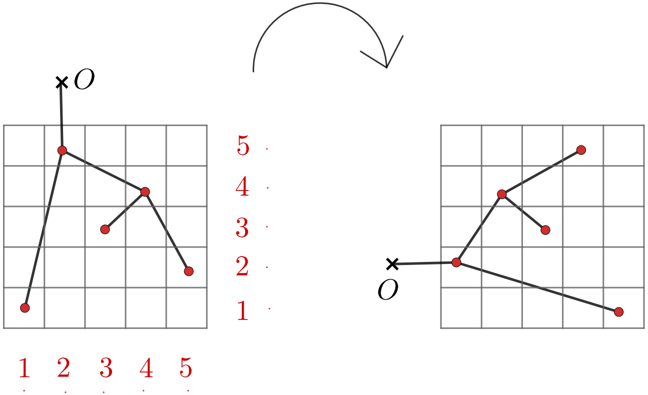
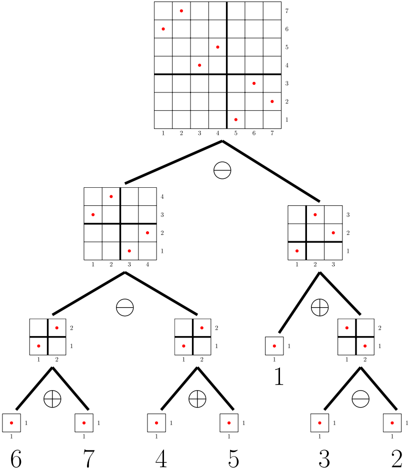


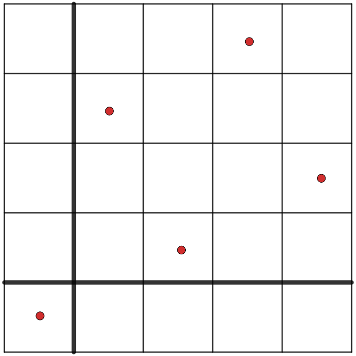
$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 5 & 3 & 4 & 2 \end{pmatrix}$$

Separable permutations



$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 7 & 4 & 5 & 1 & 3 & 2 \end{pmatrix} = ((\boxdot \oplus \boxdot) \ominus (\boxdot \oplus \boxdot)) \ominus (\boxdot \oplus (\boxdot \ominus \boxdot)).$$

Nonseparable permutation - example

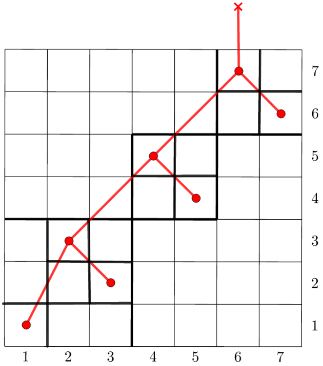


$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 2 & 5 & 3 \end{pmatrix}$$

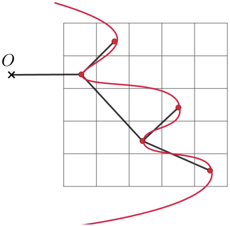
Separable tree

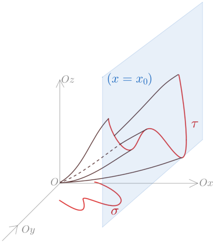
Definition

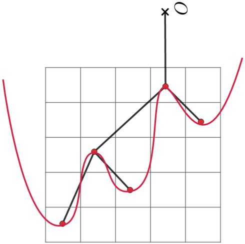
*A positive generic rooted transversal tree is **separable** if its associated permutation is separable.*



Passing to the univariate case



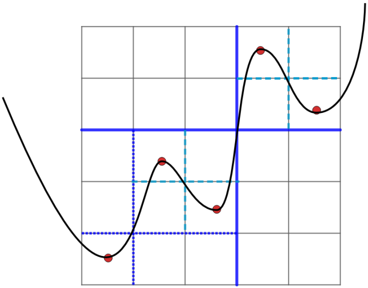


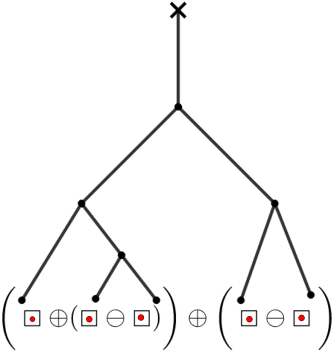


Question

Given a separable snake σ, is it possible to construct a Morse polynomial $Q : \mathbb{R} \to \mathbb{R}$ that realises σ?

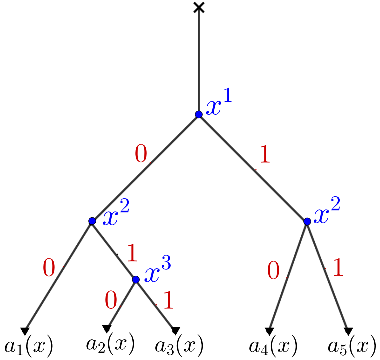
Example





$$= \left(\boxdot \oplus (\boxdot \ominus \boxdot) \right) \oplus (\boxdot \ominus \boxdot) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 2 & 5 & 4 \end{pmatrix}.$$

The contact tree



$$a_1(x) = 0,$$
$$a_2(x) = x^2,$$
$$a_3(x) = x^2 + x^3,$$
$$a_4(x) = x^1,$$
$$a_5(x) = x^1 + x^2.$$

Answer in the univariate case

Theorem ([Sor20])

Consider $m \in \mathbb{N}$ and fix a separable $(m+1)$-snake $\sigma : \{1, 2, \ldots, m+1\} \to \{1, 2, \ldots, m+1\}$ such that $\sigma(m) > \sigma(m+1)$. Construct the polynomials $a_i(x) \in \mathbb{R}[x]$ such that their contact tree is one of the binary separating trees of σ. Let $Q_x(y) \in \mathbb{R}[x][y]$ be

$$Q_x(y) := \int_0^y \prod_{i=1}^{m+1} \left(t - a_i(x)\right) \mathrm{d}t.$$

Then $Q_x(y)$ is a one variable Morse polynomial and for sufficiently small $x > 0$, the Arnold snake associated to $Q_x(y)$ is σ.

Construction of the desired bivariate polynomial f

Theorem ([Sor18])

Let σ be a separable $(m+1)$-snake, with m an even integer, $\sigma(m) > \sigma(m+1)$. Let $f \in \mathbb{R}[x,y]$ be constructed as follows:
(a) construct $Q_x(y) \in \mathbb{R}[x][y]$,

$$Q_x(y) := \int_0^y \prod_{i=1}^{m+1} \left(t - a_i(x)\right) \mathrm{d}t,$$

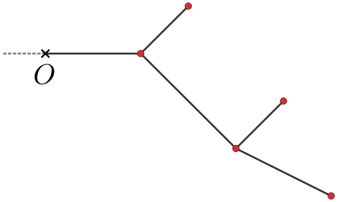
by choosing the polynomials $a_i(x) \in \mathbb{R}[x]$ such that their contact tree is one of the binary separating trees of σ.

(b) take $f(x,y) := x^2 + Q_x(y)$.
Then f has a strict local minimum at the origin and the positive asymptotic snake of f is the given σ.

Properties of f

$$f(x,y) := x^2 + \int_0^y \prod_{i=1}^{m+1} \big(t - a_i(x)\big)\mathrm{d}t.$$

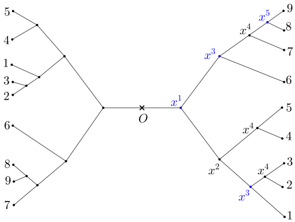
- Its positive generic asymptotic Poincaré-Reeb tree:

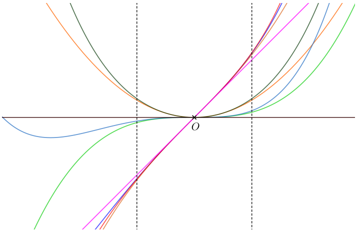


- It has a strict local minimum at the origin:

Positive-negative contact trees (one variable)[1]

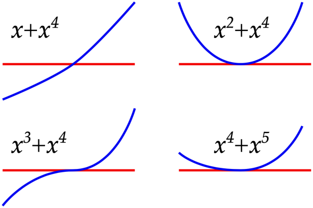
Pairwise distinct polynomials $a_i(x) \in \mathbb{R}[x]$ that pass through a common zero at the origin





[1]É. Ghys - A singular mathematical promenade, 2017

Positive-negative contact trees (one variable)[3]

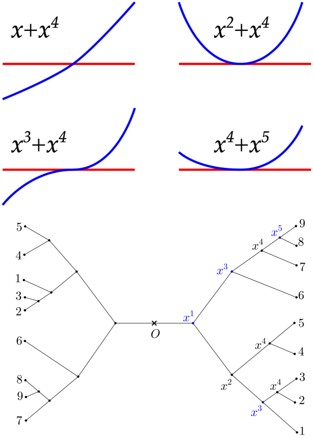


[2]

[2]Picture from [Ghy17]

[3]É. Ghys - A singular mathematical promenade, 2017

Positive-negative contact trees (one variable)[3]

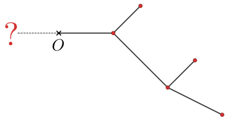


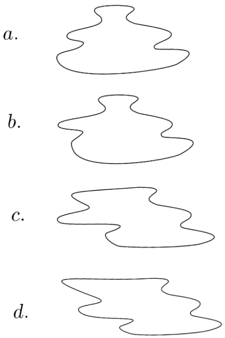
[2]

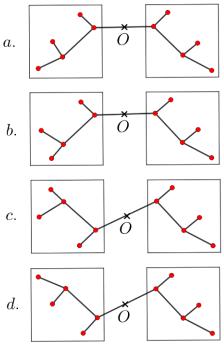
[2]Picture from [Ghy17]

[3]É. Ghys - A singular mathematical promenade, 2017

Flip-Flop Algorithm







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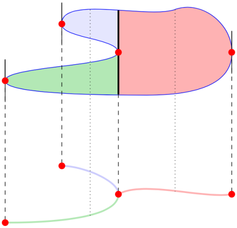
General context



Vertical planes

–> motivation: after the collapsing procedure, we end up with a new vertical plane

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–> a pair $(\mathcal{P}, \pi)$ such that $\mathcal{P}$ is a topological space homeomorphic to $\mathbb{R}^2$, endowed with an orientation, and $\pi : \mathcal{P} \to \mathbb{R}$ is a locally trivial topological fibration

Vertical planes

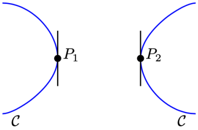
–> motivation: after the collapsing procedure, we end up with a new vertical plane

–> a pair $(\mathcal{P}, \pi)$ such that $\mathcal{P}$ is a topological space homeomorphic to $\mathbb{R}^2$, endowed with an orientation, and $\pi : \mathcal{P} \to \mathbb{R}$ is a locally trivial topological fibration

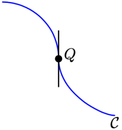
–> the **canonical affine vertical plane** is $(\mathbb{R}^2, x : \mathbb{R}^2 \to \mathbb{R})$

The topological critical set

$\Sigma_{\text{top}}(\mathcal{C})$: points $p \in \mathcal{C}$ in whose neighborhoods the restriction $\pi_{|_{\mathcal{C}}}$ is not a local homeomorhism onto its image.



Counterexample:



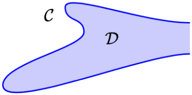
Domain of finite type

$(\mathcal{P}, \pi)$ is a vertical plane
$\mathcal{D} \subset \mathcal{P}$ is a closed subset homeomorphic to a surface with non-empty boundary $\mathcal{C}$

Definition

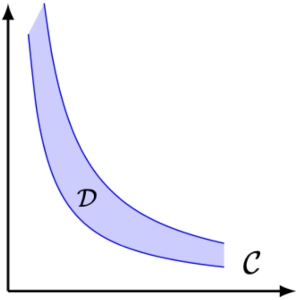
We say that $\mathcal{D}$ is a ***domain of finite type*** *in $(\mathcal{P}, \pi)$ if:*

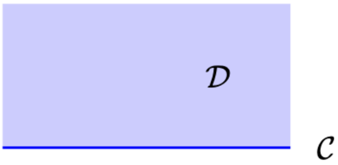
1. *the restriction $\pi_{|_{\mathcal{D}}} : \mathcal{D} \to \mathbb{R}$ is proper;*
2. *the topological critical set $\Sigma_{top}(\mathcal{C})$ is finite.*



A domain of finite type is **generic** if no two topological critical points of its boundary lie on the same vertical line.

Domains which are not of finite type:





Vertical equivalence

X and X' subsets of the vertical planes $(\mathcal{P}, \pi)$, resp. $(\mathcal{P}', \pi')$.

Definition

We say that $X \approx_v X'$, if there exist orientation preserving homeomorphisms $\Phi : \mathcal{P} \to \mathcal{P}'$ and $\psi : \mathbb{R} \to \mathbb{R}$ such that $\Phi(X) = X'$ and the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{\Phi} & \mathcal{P}' \\ {\scriptstyle \pi}\big\downarrow & & \big\downarrow{\scriptstyle \pi'} \\ \mathbb{R} & \xrightarrow[\psi]{} & \mathbb{R} \end{array}$$

Vertical equivalence

Consider the canonical affine vertical plane $(\mathbb{R}^2, x)$.

The vertical equivalence **preserves the horizontal order**:

if $x(P_i) < x(P_j)$ and $P_i' = \Phi(P_i)$, $P_j' = \Phi(P_j)$,
then $x(P_i') < x(P_j')$.

–> important for topological critical points.

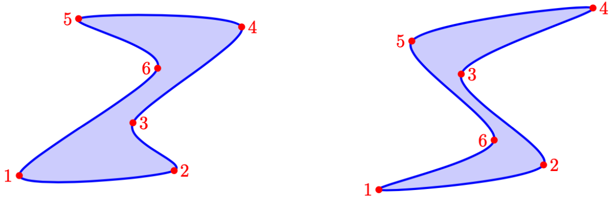
Complete invariant

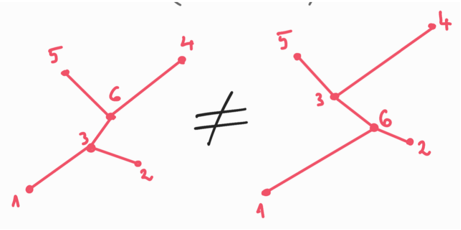
Proposition

Let $\mathcal{D}$ and $\mathcal{D}'$ be compact connected domains of finite type in vertical planes, with Poincaré–Reeb graphs G and G'. Assume that both are generic. Then:

$$\mathcal{D} \approx_v \mathcal{D}' \iff G \approx_v G'.$$

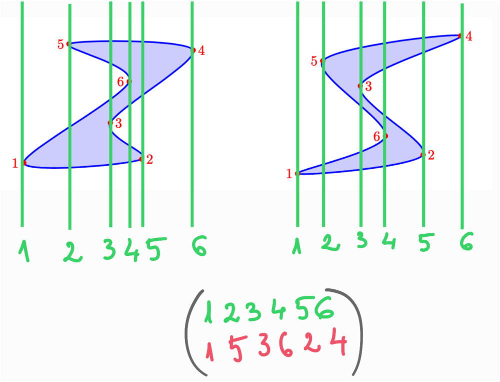
Domains which are not vertically equivalent





Other invariants?

Two generic real algebraic domains homeomorphic to discs with the same permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 5 & 3 & 6 & 2 & 4 \end{pmatrix}$, but which are not vertically equivalent.



Invariance of the Euler characteristic

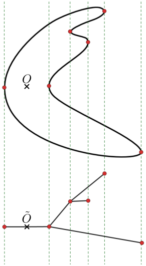
Proposition

Let $\mathcal{D}$ be a compact domain of finite type in a vertical plane. Then $\mathcal{D}$ and its Poincaré–Reeb graph have the same number of connected components and the same Euler characteristic.

Idea of the proof: integration with respect to the Euler characteristic.

Proposition

If $\mathcal{D} \subset (\mathcal{P}, \pi)$ is (homeomorphic to) a disk, then the Poincaré–Reeb graph of $\mathcal{D}$ is a tree.



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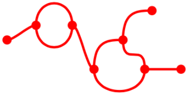
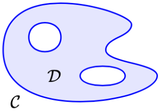
Main result

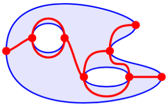
Theorem (Bodin, Popescu-Pampu, Sorea, 2022)

Any ***compact connected generic transversal*** *graph can be realized as a Poincaré–Reeb graph of an algebraic domain.*[a]

[a] [BPPS23], https://arxiv.org/pdf/2207.06871.pdf







Strategy of the proof

- Step 1: we realize the generic transversal graph G as a Poincaré–Reeb graph of a finite type domain defined by a smooth function;

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- Step 1: we realize the generic transversal graph G as a Poincaré–Reeb graph of a finite type domain defined by a smooth function;

- Step 2: we use a Weierstrass-type theorem that approximates any smooth function by a polynomial function;

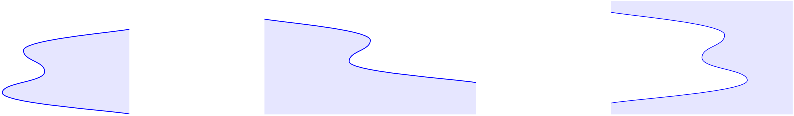
- Step 3: we adapt this Weierstrass-type theorem in order to control vertical tangents, and we realize G as the Poincaré–Reeb graph of a generic finite type algebraic domain.

Domains of weakly finite type

Definition

We say that $\mathcal{D}$ is a ***domain of weakly finite type*** *if:*

1. *the restriction $\pi_{|_{\mathcal{C}}} : \mathcal{C} \to \mathbb{R}$ is proper;*
2. *the topological critical set $\Sigma_{top}(\mathcal{C})$ is finite.*



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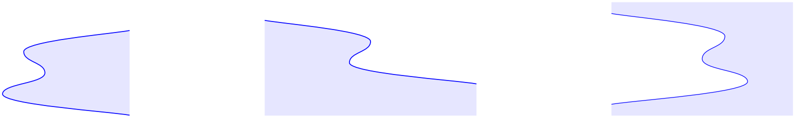
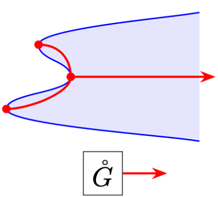
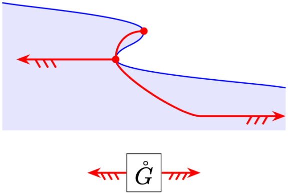


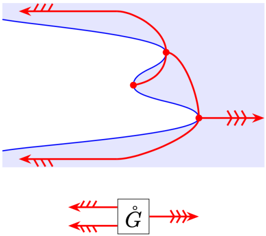
A domain of weakly finite type is called **generic** if no two topological critical points of $\mathcal{C}$ lie on the same vertical line.

Non-compact Poincaré–Reeb graphs

When $\mathcal{C}$ is homeomorphic to a line, we distinguish three cases, depending on the position of $\mathcal{D}$ and of the branches of $\mathcal{C}$.







Algebraic realization - non-compact & simply connected

Theorem (Bodin, Popescu-Pampu, Sorea, 2022)

Let G be a connected, non-compact, generic, transversal tree. Let G' be the compact tree obtained from G. If G' can be realized by a connected real algebraic curve, then G can be realized as the Poincaré–Reeb graph of a simply connected, non-compact algebraic domain in $(\mathbb{R}^2, x)$.

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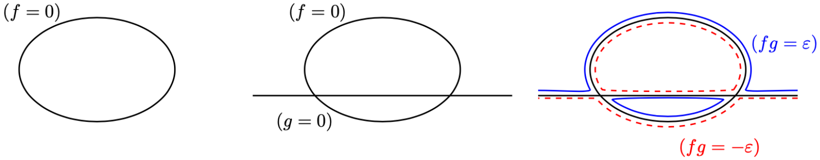
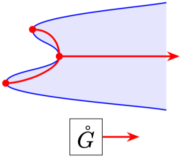
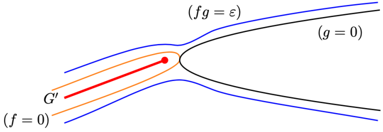
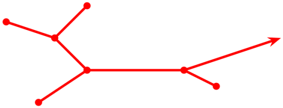
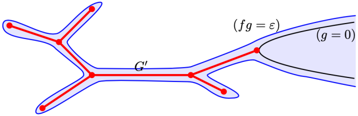


Strategy of proof - Case A

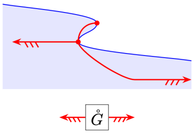


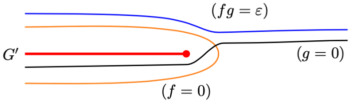


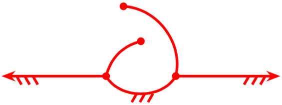


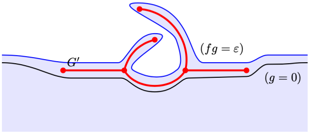


Strategy of proof - Case B



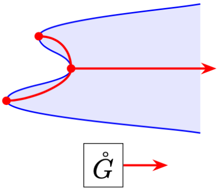


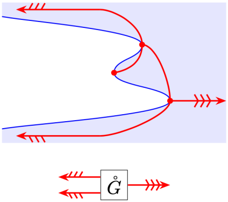




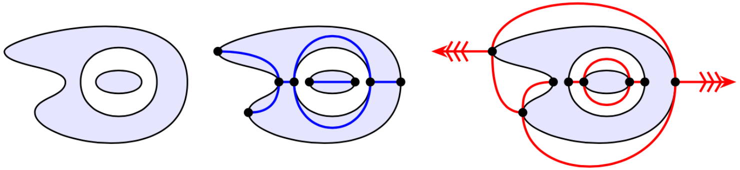
Strategy of proof - Case C

Case C is the complement of Case A:





Interior and exterior graphs of a domain of weakly finite type



Proposition

The interior graph G of a domain $\mathcal{D}$ of weakly finite type determines its exterior graph G^c.

Thank you!

Bibliography:

Arnaud Bodin, Patrick Popescu-Pampu, and Miruna-Ştefana Sorea. "Poincaré-Reeb graphs of real algebraic domains". In: *Revista Matemática Complutense* (2023). url: https://link.springer.com/article/10.1007/s13163-023-00469-y (cit. on p. 68).

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