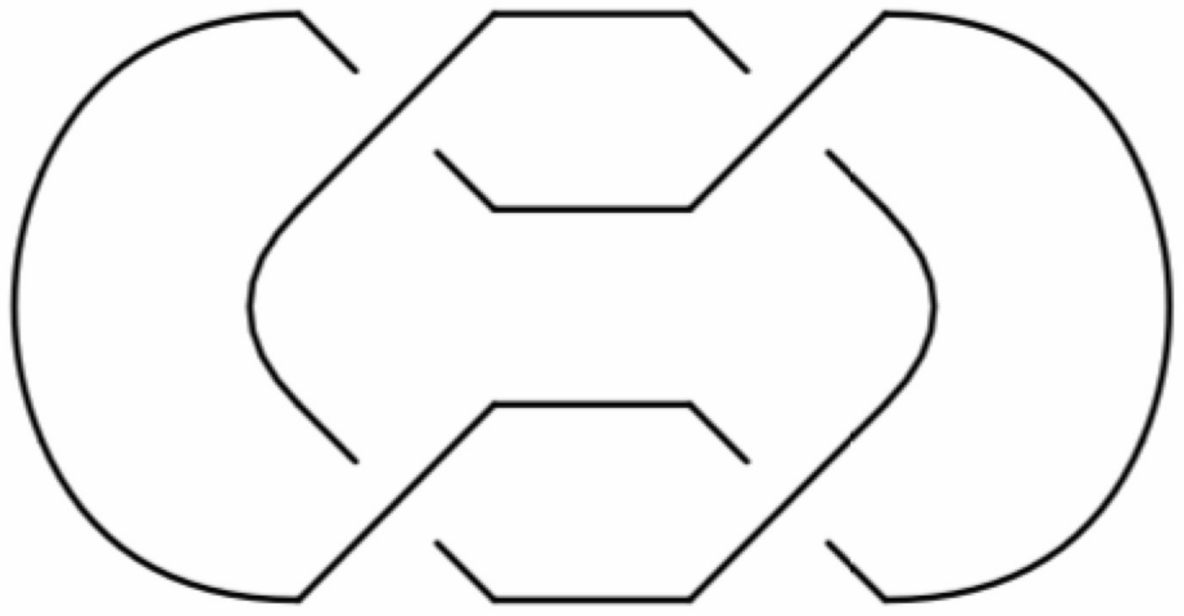
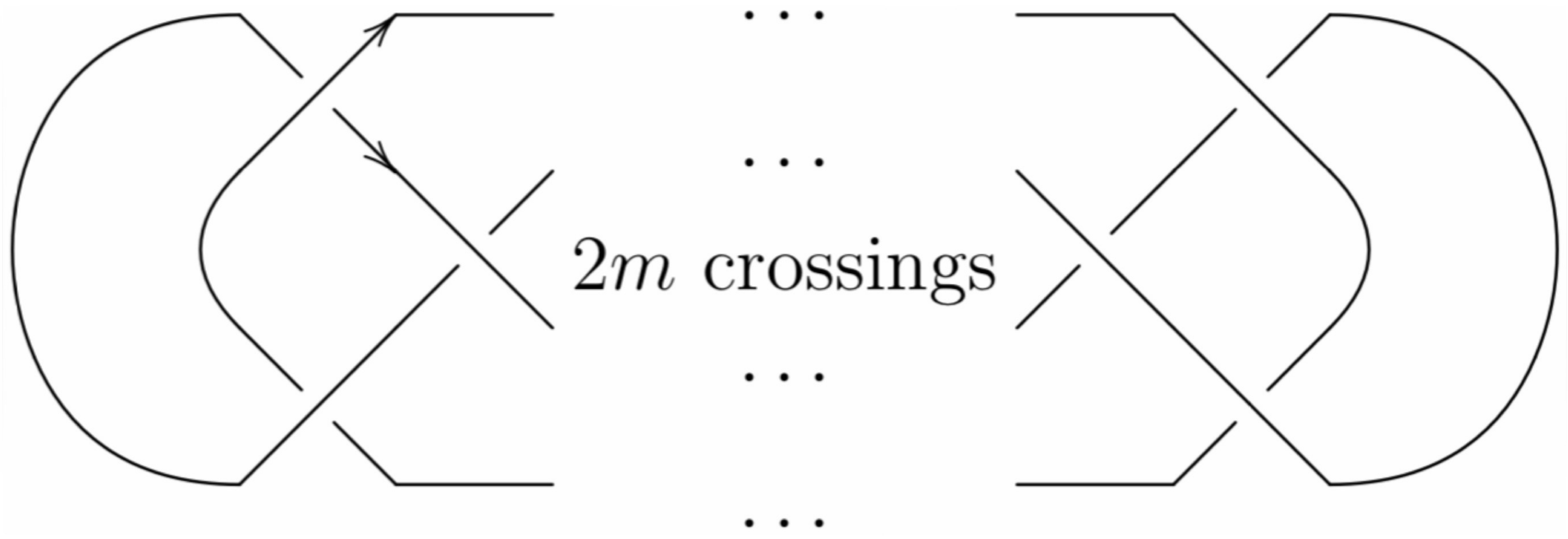
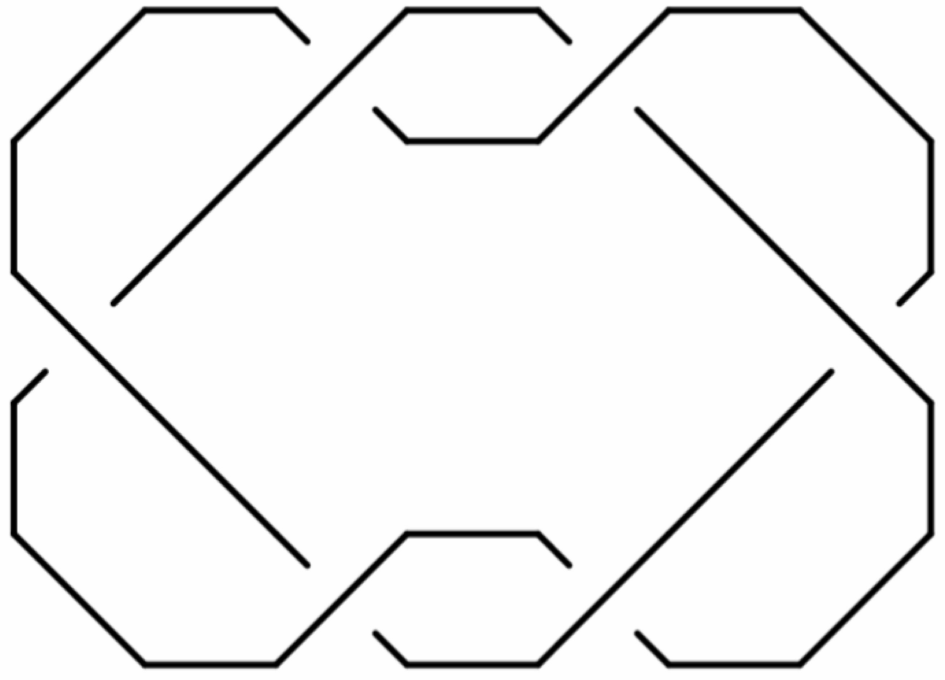
The Iwasawa invariants of $\mathbb{Z}_p^d$-covers of links.







Sohei Tateno joint work with Jun Ueki 2023/05/30

Contents

Fix a prime number p.

Notation

For an abelian group G whose p-torsion subgroup is a finite group, let $e(G)$ denote the p-exponent of the order of the p-torsion subgroup.

Example

- $e(\mathbb{Z}/p^2\mathbb{Z}) = 2$
- $e(\mathbb{Z} \oplus \mathbb{Z}/p^3\mathbb{Z}) = 3$

Historical backgrounds

For each number field k, an abelian group called an ideal class group $Cl(k)$ is defined.
The finite number $h(k) := \#Cl(k)$ is an important algebraic invariant called the class number of k.

Theorem [Kummer, 1847]
Let ζ_p denote a p-th root of unity. If $p \neq 2$ and $p \nmid h(\mathbb{Q}(\zeta_p))$, then The Fermat Last Conjecture holds for $n = p$.

Theorem(Iwasawa's class number formula) [Iwasawa, 1959]

Let k_∞/k be a $\mathbb{Z}_p$-extension and k_{p^n} be the subfields corresponding to the subgroups $p^n\mathbb{Z}_p$ of $\mathbb{Z}_p$. Then there exist $\mu, \lambda \in \mathbb{Z}_{\geq 0}$ and $\nu \in \mathbb{Z}$, depending only on k_∞/k, such that

$$e(Cl(k_{p^n})) = \mu p^n + \lambda n + \nu$$

for sufficiently large n.

Example

$$k := \mathbb{Q}(\zeta_p) \subset \mathbb{Q}(\zeta_{p^2}) \subset \mathbb{Q}(\zeta_{p^3}) \subset \ldots \subset \bigcup_{n\geq 1} \mathbb{Q}(\zeta_{p^n}) =: k_\infty.$$

A closed connected orientable 3-manifold M is called a rational homology 3-sphere($\mathbb{Q}HS^3$) if $H_i(M, \mathbb{Q}) \simeq H_i(S^3, \mathbb{Q})$ for all $i \geq 0$.

Theorem [Hillman-Matei-Morishita, 2006], [Kadokami-Mizusawa, 2008], [Ueki, 2017]

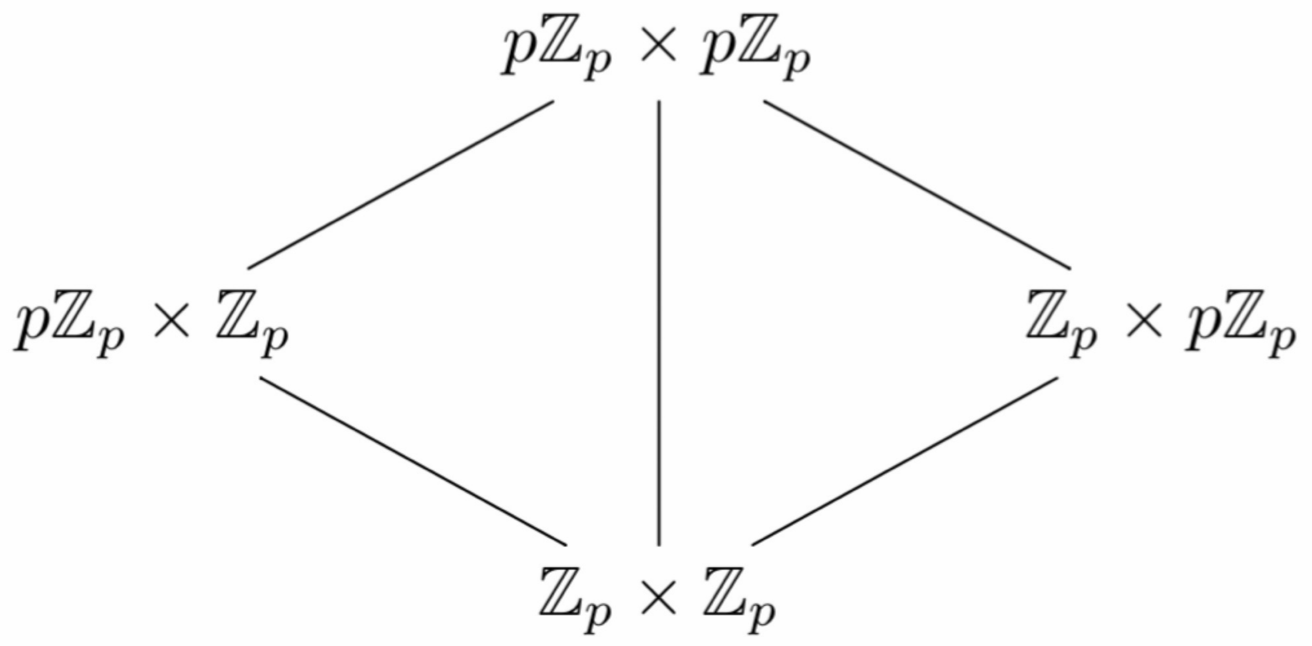
Let L be a link in a $\mathbb{Q}HS^3$ M. Let $M_{p^n} \to M$ be a compatible system of $\mathbb{Z}/p^n\mathbb{Z}$-covers branched along L. Suppose every M_{p^n} is a $\mathbb{Q}HS^3$. Then there exist $\mu, \lambda \in \mathbb{Z}_{\geq 0}$ and $\nu \in \mathbb{Z}$, depending only on $M_{p^n} \to M$ and p, such that

$e(H_1(M_{p^n})) = \mu p^n + \lambda n + \nu$ for sufficiently large n.

Theorem [Cuoco-Monsky, 1981]

Let k_∞/k be a $\mathbb{Z}_p^d$-extension and k_{p^n} be the subfields corresponding to the subgroups $(p^n\mathbb{Z}_p)^d$ of $\mathbb{Z}_p^d$. Then there exist some $\mu, \lambda \in \mathbb{Z}_{\geq 0}$, depending only on k_∞/k, such that

$$e(Cl(k_{p^n})) = (\mu p^n + \lambda n + O(1))p^{(d-1)n},$$



where O is the Bachmann-Landau notation.

Our main results

Let (M, L) be a pair of a $\mathbb{Q}HS^3$ and a link. Put $X := M - N(L)$, where $N(L)$ is an open tubular neighbourhood of L.

Let $X_\infty \to X$ be a $\mathbb{Z}^d$-cover. Let X_n be the subcovers corresponding to $(n\mathbb{Z})^d$. Let M_n be the Fox completions of X_n.

Let $W := \{\zeta \in \mathbb{C} \mid \zeta^{p^n} = 1$ for some $n \geq 0\}$ and let $\Delta(t_1, \ldots, t_d) \in \mathbb{Z}[t_1^{\pm 1}, \ldots, t_d^{\pm 1}]$ denote the Alexander polynomial of $X_\infty \to X$, which is corresponding to characteristic polynomials in Iwasawa theory.

Main result 1

Suppose that $\Delta(t_1, \ldots, t_d)$ does not vanish on $(W \backslash \{1\})^d$. Suppose each M_{p^n} is a $\mathbb{Q}HS^3$. Then there exist $\mu, \lambda \in \mathbb{Z}_{\geq 0}$, depending only on $X_\infty \to X$ and p, such that

$e(H_1(M_{p^n})) = (\mu p^n + \lambda n + O(1))p^{(d-1)n}$.

M is called an integral homology 3-sphere($\mathbb{Z}HS^3$) if $H_i(M, \mathbb{Z}) \simeq H_i(S^3, \mathbb{Z})$ for all $i \geq 0$.

Main result 2

Suppose that M is a $\mathbb{Z}HS^3$, L consists of d components, and $\Delta_L(t_1, \ldots, t_d)$ does not vanish on $(W \backslash \{1\})^d$. Then there exist $\mu, \lambda \in \mathbb{Z}_{\geq 0}$ and $\mu_{d-1}, \ldots, \mu_1, \lambda_{d-1}, \ldots, \lambda_1, \nu \in \mathbb{Q}$, depending only on L and p, such that

$$e(H_1(M_{p^n})) = \mu p^{dn} + \lambda n p^{(d-1)n} + \mu_{d-1} p^{(d-1)n} + \lambda_{d-1} n p^{(d-2)n} + \ldots + \mu_1 p^n + \lambda_1 n + \nu\,.$$

for sufficiently large n.

Remark

We have

M is $\mathbb{Q}HS^3 \Longleftrightarrow H_1(M)$ is finite

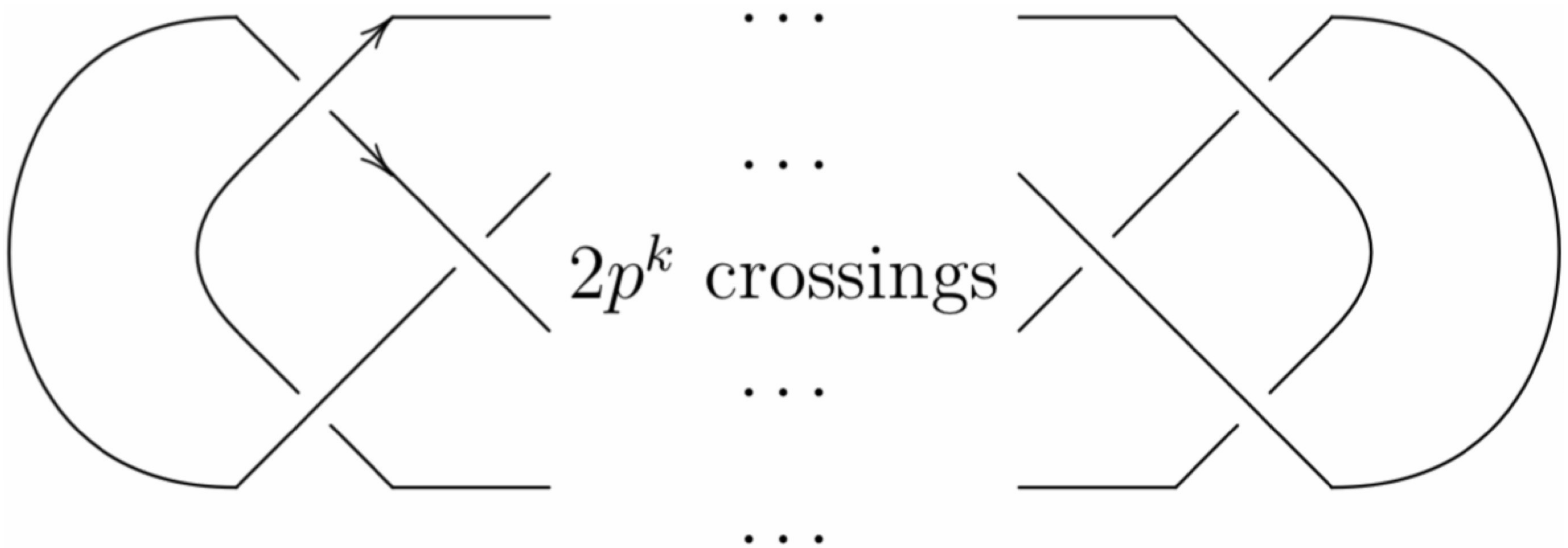
M is $\mathbb{Z}HS^3 \Longleftrightarrow H_1(M) = 0$

Hence

$S^3 \in \{\mathbb{Z}HS^3\} \subset \{\mathbb{Q}HS^3\}$.

($\mathbb{Q} \in \{$num. fields with $h(k) = 0\} \subset \{$num. fields$\}$)

<u>Example</u>



Let $d = 2$ and $M := S^3$. For the twisted Whitehead link $L := W_{2p^k}$, using a result of Porti, we obtain

$$|H_1(M_{p^n})| = p^{(kp^n+2n-2k)p^n-2n+2k}.$$

Definition of Iwasawa invariants

Let $\Lambda_{\mathbb{Z}} := \mathbb{Z}[t_1^{\pm 1}, \ldots, t_d^{\pm 1}]$ and $\Lambda := \mathbb{Z}_p[[T_1, \ldots, T_d]]$. We have an embedding $\Lambda_{\mathbb{Z}} \hookrightarrow \varprojlim_n (\mathbb{Z}/p^n\mathbb{Z})[t_1^{\mathbb{Z}/p^n\mathbb{Z}}, \ldots, t_d^{\mathbb{Z}/p^n\mathbb{Z}}] \simeq \Lambda$ sending $\Lambda_{\mathbb{Z}} \ni t_i \to 1 + T_i \in \Lambda$.

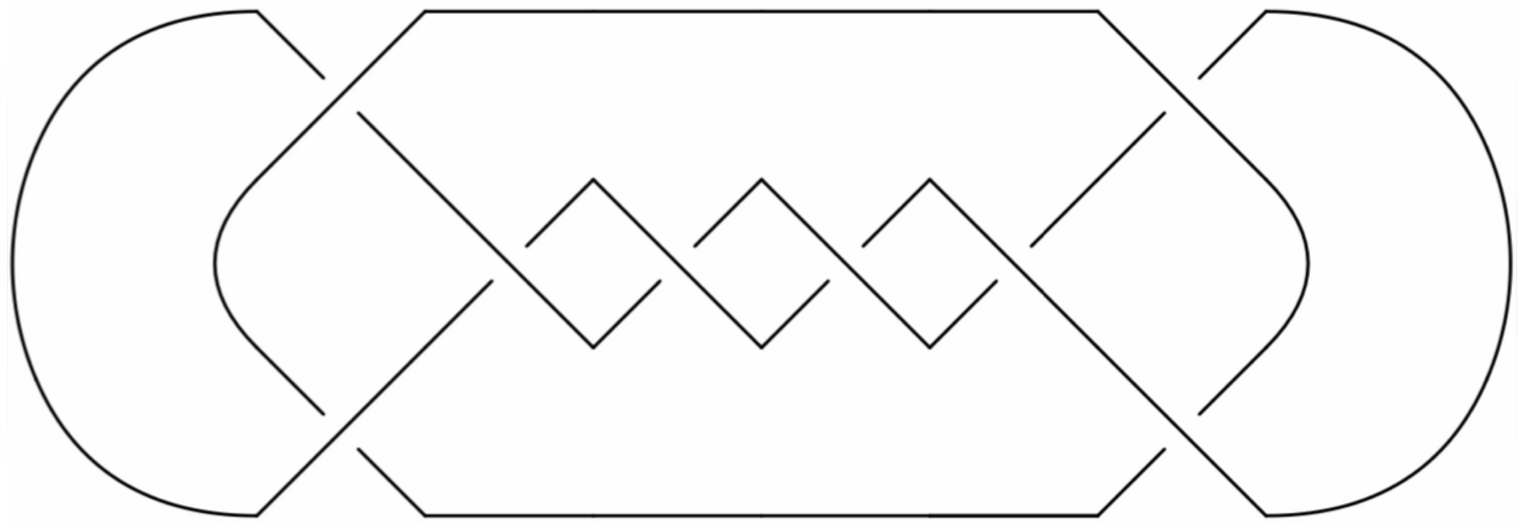
Let $\Delta(1 + T_1, \ldots, 1 + T_d) \in \Lambda$ be the Alexander polynomial. Then there exist unique $F_0 \in \Lambda$ and $\mu \in \mathbb{Z}_{\geq 0}$ such that $\Delta = p^{\mu} F_0$ and $p \nmid F_0$.

Let $\overline{F}_0 := F_0 \bmod p$ in $(\mathbb{Z}/p\mathbb{Z})[[T_1, \ldots, T_d]]$.

Let λ be the number of polynomials of the form $(\overline{1 + T_1})^{r_1} \cdots (\overline{1 + T_d})^{r_d} - 1$ with $p \nmid r_i$ $(\exists i)$ that divides $\overline{F}_0$.

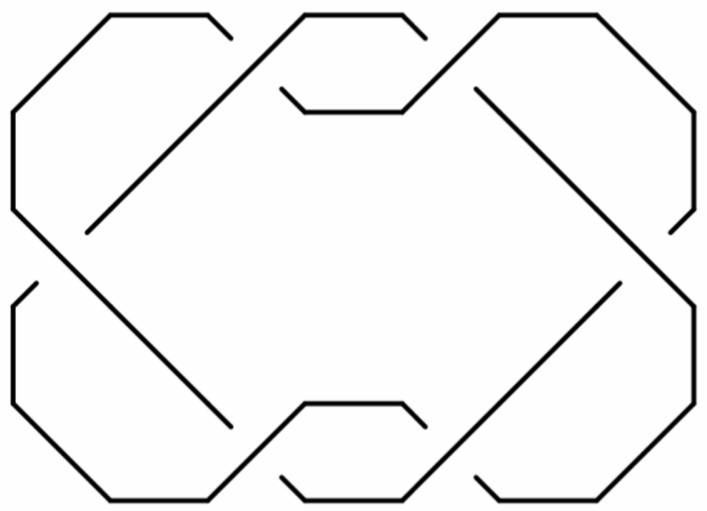
Example

If $L = 7_3^2$, then



$$\Delta_L(1+X, 1+Y) = 2XY = 2((1+X)-1)((1+Y)-1).$$

Hence $\lambda = 2$. Moreover, $\mu = 1$ if $p = 2$.



If $L = 6_1^2$, then

$$\Delta_L(1+X, 1+Y) = X^2Y^2 + 2X^2Y + 2XY^2 + X^2 + 5XY + Y^2 + 3X + 3Y + 3$$

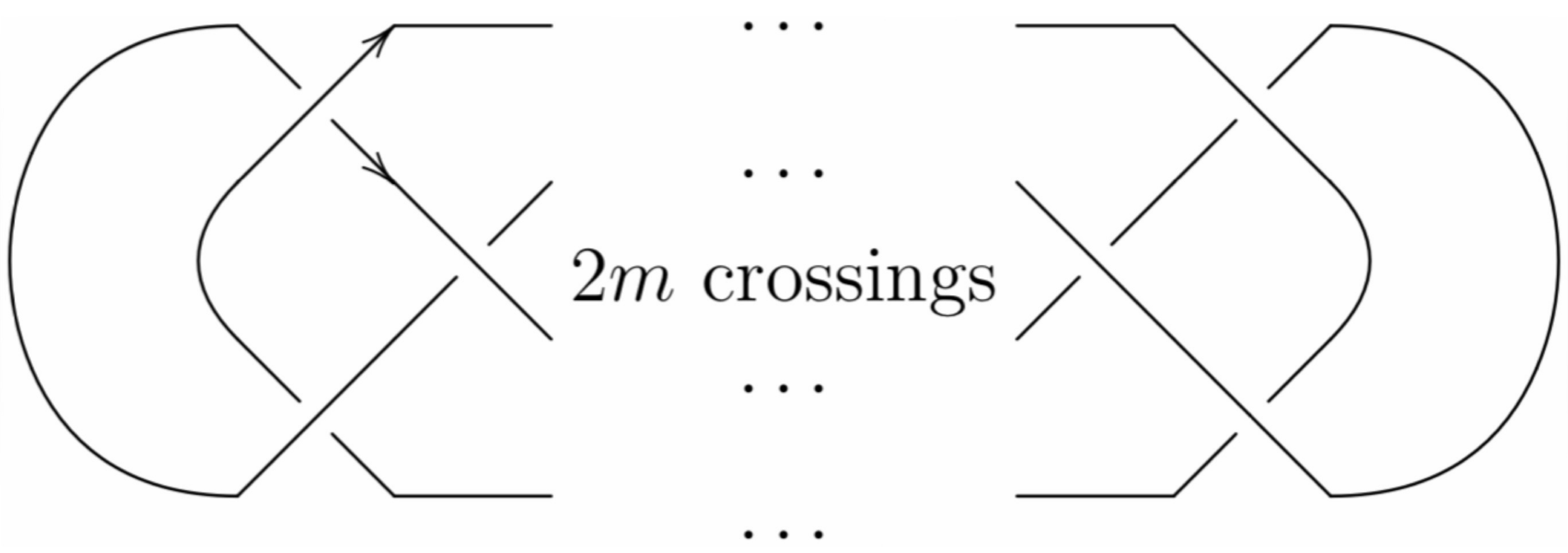
$\equiv ((1+X)(1+Y)-1)^2 \mod 3$. Hence $\lambda = 2$ if $p = 3$.

We made a complete table of μ and λ invariants.

link	Alexander polynomial of $\Delta_L(1+X,1+Y,\ldots)$	μ	λ
4^2_1	$XY+X+Y+2$	0	1 if $p=2$
5^2_1	XY	0	2
6^2_1	$X^2Y^2+2X^2Y+2XY^2+X^2+5XY+Y^2+3X+3Y+3$	0	2 if $p=3$
6^2_2	$X^2Y+XY^2+X^2+3XY+Y^2+3X+3Y+3$	0	0
6^2_3	$2XY+X+Y+2$	0	0
7^2_1	$X^2Y^2+X^2Y+XY^2+XY-X-Y-1$	0	0
7^2_2	$X^2Y^2+X^2Y+XY^2+3XY+X+Y+1$	0	0
7^2_3	$2XY$	1 if $p=2$	2
7^2_4	X^3Y+2X^2Y+2XY	0	4 if $p=2$ 2 if not
7^2_5	$X^3Y+X^3+X^2Y+3X^2+XY+3X+Y+2$	0	1 if $p=2$
7^2_6	X^3Y+X^2Y+XY	0	2
7^2_7	$X^3Y+X^3+3X^2Y+3X^2+3XY+3X+Y+2$	0	1 if $p=2$
7^2_8	XY	0	2
8^2_1	$X^3Y^3+3X^3Y^2+3X^2Y^3+3X^3Y+9X^2Y^2+3XY^3+2X^3+10X^2Y+10XY^2$ $+Y^3+7X^2+13XY+4Y^2+9X+6Y+4$	0	0
8^2_2	$X^3Y+X^2Y^2+XY^3+X^3+4X^2Y+4XY^2+Y^3+4X^2+7XY$ $+4Y^2+6X+6Y+4$	0	0
8^2_3	$2X^2Y^2+3X^2Y+3XY^2+X^2+7XY+Y^2+3X+3Y+3$	0	0
8^2_4	$X^3Y^2+X^2Y^3+2X^3Y+4X^2Y^2+2XY^3+X^3+7X^2Y+7XY^2+Y^3$ $+4X^2+10XY+4Y^2+6X+6Y+4$	0	3 if $p=2$
8^2_5	$X^2Y^2-X^2-XY-Y^2-3X-3Y-3$	0	0
8^2_6	$3XY+X+Y+2$	0	1 if $p=2$
8^2_7	$X^2Y^2-XY-X-Y-1$	0	0
8^2_8	$X^2Y^2+XY+X+Y+1$	0	0
8^2_9	$-X^3-2X^2Y-X^2+3X+Y+2$	0	1 if $p=2$
8^2_{10}	X^3Y	0	4
8^2_{11}	$-X^3Y+X^3-X^2Y+3X^2-XY+3X+Y+2$	0	1 if $p=2$
8^2_{12}	X^3Y	0	4
8^2_{13}	X^3Y-X^2Y-XY	0	2
8^2_{14}	$X^3Y+X^3-X^2Y+3X^2-XY+3X+Y+2$	0	1 if $p=2$
8^2_{15}	XY	0	2
8^2_{16}	$-X^3-X^2+2XY+3X+Y+2$	0	1 if $p=2$
9^2_1	$X^3Y^3+2X^3Y^2+2X^2Y^3+X^3Y+4X^2Y^2+XY^3+X^2Y+XY^2-X^2$ $-2XY-Y^2-3X-3Y-2$	0	0
9^2_2	$X^3Y+X^2Y^2+XY^3+2X^2Y+2XY^2-X^2-XY-Y^2-3X-3Y$ -2	0	0
9^2_3	$2X^2Y^2+2X^2Y+2XY^2+3XY-X-Y-1$	0	0
9^2_4	$X^3Y^2+X^2Y^3+X^3Y+5X^2Y^2+XY^3+5X^2Y+5XY^2+5XY$	0	2
9^2_5	$X^3Y+2X^2Y^2+XY^3+4X^2Y+4XY^2+4XY$	0	4 if $p=2$ 2 if not
9^2_6	$-X^3Y^2-X^2Y^3-X^3Y-3X^2Y^2-XY^3-2X^2Y-2XY^2+X^2+XY$ $+Y^2+3X+3Y+2$	0	1 if $p=2$
9^2_7	$-X^3Y^2-X^2Y^3-X^3Y-2X^2Y^2-XY^3-X^2Y-XY^2+X^2+2XY$ $+Y^2+3X+3Y+2$	0	1 if $p=2$
9^2_8	$2X^2Y+2XY^2+3XY-X-Y-1$	0	0
9^2_9	$X^3Y^2-X^2Y^3+X^3Y-3X^2Y^2-XY^3-3X^2Y-3XY^2-2X^2-3XY$ $-2X$	0	1
9^2_{10}	$3XY$	1 if $p=3$	2
9^2_{11}	$2X^2Y^2+X^2Y+XY^2+XY-X-Y-1$	0	0
9^2_{12}	$X^2Y^2-X^2Y-XY^2-XY+X+Y+1$	0	0
9^2_{13}	$X^5Y+4X^4Y+7X^3Y+6X^2Y+3XY$	0	4 if $p=3$ 2 if not
9^2_{14}	$X^5+2X^4Y+5X^4+6X^3Y+10X^3+8X^2Y+10X^2+4XY+5X$ $+Y+2$	0	1 if $p=2$
9^2_{15}	$2X^3Y+3X^2Y+3XY$	0	4 if $p=3$ 2 if not
9^2_{16}	$2X^3+4X^2Y+5X^2+3XY+3X+Y+2$	0	1 if $p=2$
9^2_{17}	$2X^3+3X^2Y+5X^2+2XY+3X+Y+2$	0	1 if $p=2$
9^2_{18}	$2X^3Y+2X^2Y+2XY$	1 if $p=2$	2
9^2_{19}	$X^4Y+X^3Y^2+5X^3Y+2X^2Y^2+X^3+8X^2Y+2XY^2+2X^2+6XY$ $+2X+Y+1$	0	0
9^2_{20}	$X^4+2X^3Y+2X^2Y^2+4X^3+7X^2Y+2XY^2+7X^2+6XY+Y^2$ $+6X+3Y+3$	0	0
9^2_{21}	$X^4Y^2+X^4Y+3X^3Y^2+5X^3Y+4X^2Y^2+X^3+8X^2Y+2XY^2+2X^2$ $+6XY+2X+Y+1$	0	0
9^2_{22}	$X^4Y+2X^3Y^2+X^4+5X^3Y+4X^2Y^2+4X^3+10X^2Y+3XY^2+7X^2$ $+8XY+Y^2+6X+3Y+3$	0	0
9^2_{23}	$X^3Y+2X^2Y^2+XY^3+3X^2Y+3XY^2-X^2-Y^2-3X-3Y-2$	0	1 if $p=2$
9^2_{24}	$3X^2Y^2+3X^2Y+3XY^2+X^2+7XY+Y^2+3X+3Y+3$	0	2 if $p=3$
9^2_{25}	X^3Y-2X^2Y-2XY	0	4 if $p=2$ 2 if not
9^2_{26}	$X^3Y-X^3-X^2Y-X^2+XY+3X+Y+2$	0	1 if $p=2$
9^2_{27}	$2X^3Y+3X^2Y+3XY$	0	4 if $p=3$ 2 if not
9^2_{28}	$X^3Y-X^3-X^2Y-3X^2-XY-3X-Y-2$	0	1 if $p=2$
9^2_{29}	$-2X^4Y-5X^3Y-6X^2Y-3XY+1$	0	0
9^2_{30}	$-X^3Y+2X^3+X^2Y+5X^2+3X+Y+2$	0	0
9^2_{31}	$X^5Y+3X^4Y+4X^3Y+2X^2Y+XY$	0	2
9^2_{32}	$2X^3Y+X^2Y+XY$	0	2
9^2_{33}	$2X^3Y+X^2Y+XY$	0	2
9^2_{34}	$X^4Y^2+X^4Y+2X^3Y^2+3X^3Y+2X^2Y^2+2X^2Y+XY^2-X^2-2X$ $-Y-1$	0	0
9^2_{35}	$X^4Y^2+X^4Y+2X^3Y^2+3X^3Y+2X^2Y^2+4X^2Y+XY^2+X^2+4XY$ $+2X+Y+1$	0	0
9^2_{36}	$2X^3Y+2X^2Y+2XY$	1 if $p=2$	2
9^2_{37}	$X^5Y+3X^4Y+5X^3Y+4X^2Y+2XY$	0	4 if $p=2$ 2 if not
9^2_{38}	$2X^3Y+X^2Y+X^2+2XY+3X+Y+2$	0	1 if $p=2$
9^2_{39}	$-X^4Y-2X^3Y^2-X^4-4X^3Y-4X^2Y^2-3X^3-7X^2Y-3XY^2-4X^2$ $-4XY-Y^2-2X-Y$	0	0
9^2_{40}	$X^4Y^2+2X^4Y+X^3Y^2+X^4+5X^3Y+4X^3+4X^2Y+XY^2+7X^2$ $+4XY+Y^2+6X+3Y+3$	0	2 if $p=3$
9^2_{41}	$X^3Y^3+2X^3Y^2+X^2Y^3+X^3Y+5X^2Y^2+3X^2Y+3XY^2+3XY$	0	4 if $p=3$ 2 if not
9^2_{42}	$X^4Y^2+X^4Y+X^3Y^2+2X^3Y-X^2Y-X^2-2XY-2X-Y$ -1	0	0
9^2_{43}	$X^5+5X^4+10X^3+10X^2+5X+Y+2$	0	1 if $p=2$
9^2_{44}	X^3Y+2X^2Y+2XY	0	4 if $p=2$ 2 if not
9^2_{45}	$2X^3+5X^2-XY+3X+Y+2$	0	1 if $p=2$
9^2_{46}	$2XY$	1 if $p=2$	2
9^2_{47}	XY	0	2
9^2_{48}	$2X^3+X^2Y+5X^2+3X+Y+2$	0	1 if $p=2$
9^2_{49}	$X^4+4X^3+X^2Y+7X^2+2XY+Y^2+6X+3Y+3$	0	2 if $p=3$
9^2_{50}	$-X^2Y-XY+Y^2+X+Y+1$	0	0
9^2_{51}	$X^4Y+X^4+3X^3Y+4X^3+4X^2Y+XY^2+7X^2+4XY+Y^2$ $+6X+3Y+3$	0	0
9^2_{52}	$X^2Y^2+X^2Y+XY-Y^2-X-Y-1$	0	0
9^2_{53}	$X^2Y^2+X^3+2X^2Y+2XY^2+Y^3+4X^2+5XY+4Y^2+6X$ $+6Y+4$	0	2 if $p=2$
9^2_{54}	$X^2Y+XY^2+XY-X-Y-1$	0	0
9^2_{55}	X^3Y+X^2Y+XY	0	2
9^2_{56}	X^3Y+X^2Y+XY	0	2
9^2_{57}	$X^3Y+2X^2Y+X^2+3XY+3X+Y+2$	0	0
9^2_{58}	$X^3Y+X^2Y+X^2+2XY+3X+Y+2$	0	0
9^2_{59}	$X^5Y+X^5+5X^4Y+5X^4+9X^3Y+10X^3+8X^2Y+10X^2+4XY$ $+5X+Y+2$	0	0
9^2_{60}	$X^3Y+2X^3+2X^2Y+5X^2+XY+3X+Y+2$	0	0
9^2_{61}	$X^3Y^2+2X^3Y+3X^2Y^2+XY^3+X^3+6X^2Y+6XY^2+Y^3+4X^2$ $+9XY+4Y^2+6X+6Y+4$	0	2 if $p=2$
6^3_1	$XY+XZ+YZ+X+Y+Z$	0	0
6^3_2	$-XYZ$	0	3
6^3_3	$-XYZ-XY-XZ-YZ-X-Y-Z$	0	1
7^3_1	$-XYZ+YZ-X+Y+Z$	0	0
8^3_1	$-X^2Y^2Z-2X^2YZ-XY^2Z-X^2Y+XY^2-X^2Z-3XYZ-X^2+Y^2$ $-2XZ-2X+2Y-Z$	0	0
8^3_2	$X^2Y^2+X^2YZ+XY^2Z+2X^2Y+2XY^2+X^2Z+2XYZ+Y^2Z+X^2$ $+4XY+Y^2+2XZ+2YZ+2X+2Y+Z$	0	0
8^3_3	$XYZ-XY-XZ-YZ-X-Y-Z$	0	1 if $p=2$
8^3_4	$X^3Y+2X^2YZ+X^3+3X^2Y+X^2Z+2XYZ+4X^2+2XY+2XZ$ $+4X$	0	3 if $p=2$ 1 if not
8^3_5	$X^2Y^2Z+X^2YZ+XY^2Z+2XYZ$	0	4 if $p=2$ 3 if not
8^3_6	$X^2Y^2Z+X^2YZ+XY^2Z+3XYZ+XZ+YZ+Z$	0	1
8^3_7	$-X^2Y^2Z-X^2Y^2-2X^2YZ-2XY^2Z-2X^2Y-2XY^2-X^2Z-4XYZ-Y^2Z$ $-X^2-4XY-Y^2-2XZ-2YZ-2X-2Y-Z$	0	1
8^3_8	$XY^2Z-X^2Y+XY^2+XYZ+Y^2Z-X^2+Y^2+2YZ-2X$ $+2Y+Z$	0	0
8^3_9	XYZ	0	3
8^3_{10}	$X^3YZ+X^3Y+X^3Z+3X^2YZ+X^3+3X^2Y+3X^2Z+2XYZ+4X^2$ $+2XY+2XZ+4X$	0	3 if $p=2$ 1 if not
9^3_1	$X^2Y^2Z+X^2YZ+X^2Y-XY^2-XZ-YZ-Z$	0	0
9^3_2	$X^2Y^2Z+X^2YZ+X^2Y-XY^2+2XYZ+XZ+YZ+Z$	0	0
9^3_3	$-X^3YZ-X^2YZ-X^3+X^2Y+X^2Z-XYZ-X^2+XY+XZ$ $+YZ-X+Y+Z$	0	0
9^3_4	$X^3Y+X^3Z+2X^2YZ+X^3+2X^2Y+2X^2Z+2XYZ+X^2+2XY$ $+2XZ+YZ+X+Y+Z$	0	0
9^3_5	$X^2YZ+XY^2Z+2XYZ-Y^2Z+X^2-Y^2-2YZ+2X-2Y$ $-Z$	0	0
9^3_6	$X^2Y^2Z+X^2YZ+X^2Y-XY^2+XYZ-Y^2Z+X^2-Y^2-2YZ$ $+2X-2Y-Z$	0	0
9^3_7	$2XYZ-YZ+X-Y-Z$	0	1 if $p=2$
9^3_8	$X^3Y+2X^2YZ+2X^2Y+XYZ$	0 0	3 if $p=2,3$ 2 if not
9^3_9	$X^3YZ+X^2YZ+XYZ$	0	3
9^3_{10}	X^2Y^2Z	0	5
9^3_{11}	$X^2Y^2Z-8XYZ-Y^2Z+X^2-Y^2-8XZ-10YZ+2X-2Y$ $-9Z$	0	0
9^3_{12}	X^3YZ	0	5
9^3_{13}	$X^2Y^2Z+X^2Y^2+X^2YZ+XY^2Z+X^2Y+XY^2-XZ-YZ-Z$	0	0
9^3_{14}	$X^2Y^2Z+X^2Y^2+X^2YZ+XY^2Z+X^2Y+XY^2+2XYZ+XZ+YZ$ $+Z$	0	0
9^3_{15}	$X^3Y+X^3Z+X^3+2X^2Y+2X^2Z+X^2+2XY+2XZ+YZ$ $+X+Y+Z$	0	0
9^3_{16}	$-2X^2YZ+X^3-X^2Y-X^2Z-2XYZ+X^2-XY-XZ-YZ$ $+X-Y-Z$	0	0
9^3_{17}	$-X^3+X^2Y+X^2Z-X^2+XY+XZ+YZ-X+Y$ $+Z$	0	0
9^3_{18}	$-XYZ$	0	3
9^3_{19}	$-X^3YZ-X^3Y-2X^2YZ-2X^2Y-XYZ$	0	3
9^3_{20}	$-X^3Z-X^2YZ-2X^2Z$	0	4 if $p=2$ 3 if not
8^4_1	$-WXY-WXZ-WYZ-XYZ-WY-XY-WZ-XZ$	0	0
8^4_2	$-WXZ-WYZ-WY+XY-WZ-XZ$	0	0
8^4_3	$WXYZ+WXY+WXZ+WYZ+XYZ+WY+XY+WZ+XZ$	0	2
8^4_4	$WXYZ-WXZ-WYZ-WY+XY-WZ-XZ$	0	0

Let W_{2m} denote twisted Whitehead links in S^3. By a calculation using the potential functions of links, we obtain

$$\Delta_{W_{2m}}(x, y) = m(x-1)(y-1)\,.$$



i.e.,

$$\Delta_{W_{2m}}(1+X, 1+Y) = mXY\,.$$

This yields $\mu_{W_{2pk}} = k$. In particular, we have

Main result 3

Suppose $d = 2$. Then, for arbitrary $k \geq 0$, there exists a link L such that $\mu_L = k$.

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