About some Steiner trees

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**Problem**: to connect a (usually finite) set of points by the shortest connected set (usually at the plane):

We denote by $\mathcal{H}^1$ the linear Hausdorff measure (roughly speaking, length).

**Problem (Euclidean Steiner problem)**

Let $C$ be a compact subset of $\mathbb{R}^d$. To find a closed $S$ such that $S \cup C$ is connected and $\mathcal{H}^1(S)$ is minimal.

If $C$ is totally disconnected then $S$ should be connected.
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- only two variants of neighbourhoods for points from $S \setminus C$. 

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- $S$ exists;
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  - a regular tripod ($x$ is a **branching point**);
  - a segment; $x$ is an inner point.
- $S$ contains at most countable number of branching points

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Theorem (Paolini–Stepanov–T, 2015; Cherkashin–T. 2023; Paolini–Stepanov 2023)

There is a compact planar set $M$ for which the unique solution of the Steiner problem has infinite number of triple points.

$M$, $\Sigma$ are self-similar fractals with sufficiently small scale.

Figure: Indecomposable Steiner tree with infinite number of triple points
We say that full Steiner tree $S$ connecting finite set $C = \{A_1 \ldots A_n\}$ has \textit{adding property} if there exists such $\varepsilon > 0$ that $\bigcup_{i=1}^{n} [A_iB_i] \cup S$ is Steiner tree for $\{B_1 \ldots B_n\}$ where $|A_iB_i| = \varepsilon$ and $B_i$ belongs to the ray beginning from the segment of $S$ incident $A_i$.

\textbf{Theorem}

\textit{Cherkashin, T., 2022; reformulated} Let $St$ be a Steiner tree for terminals $A = (A_1, ..., A_m)$, $A_i \in \mathbb{R}^n$ such that every Steiner tree for an $n$-tuple in the closed $2r$-neighbourhood of $A$ has the same topology as $St$ for some positive $r$. Then $St$ has adding property.

\textbf{Example: the tripod}

Usually the condition holds:

\textbf{Theorem (Basok, Cherkashin, T., 2022)}

For $m \geq 4$ the set of $m$ terminals with non unique Steiner trees has the Hausdorff dimension $2m - 1$.

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Theorem (Cherkashin, T., 2022)

Let $S_t$ be a Steiner tree for terminals $A = (A_1, ..., A_m)$, $A_i \in \mathbb{R}^n$ such that every Steiner tree for an $n$-tuple in the closed $2r$-neighbourhood of $A$ has the same topology as $S_t$ for some positive $r$. Then $S_t$ has adding property.

It turns out that a Steiner tree for the vertices of a square does not have this property:
Figure: The left part contains two Steiner trees connecting vertices of a square; the right part provides an example of a Steiner tree with an infinite number of branching points $y_i$, $i \geq 1$. 
Let \((X, \rho)\) be a metric space. For any subset \(U \subset X\), let \(\text{diam } U\) denote its diameter, that is \(\text{diam } U := \sup\{\rho(x, y) : x, y \in U\}\), \(\text{diam } \emptyset := 0\).

Let \(S\) be any subset of \(X\), and \(\delta > 0\) a real number. Define

\[
H^d_\delta(S) = \inf \left\{ \sum_{i=1}^{\infty} (\text{diam } U_i)^d : \bigcup_{i=1}^{\infty} U_i \supseteq S, \text{diam } U_i < \delta \right\}
\]

where the infimum is over all countable covers of \(S\) by sets \(U_i \subset X\) satisfying \(\text{diam } U_i < \delta\).

Let

\[
H^d(S) := \sup_{\delta > 0} H^d_\delta = \lim_{\delta \to 0^+} H^d_\delta(S).
\]