About some Steiner trees

Yana Teplitskaya

Université Paris-Saclay

Algebraic and geometric methods of analysis
May 29 - June 1, 2023

Statement of Steiner problem

<u>Problem</u>: to connect a (usually finite) set of points by the shortest connected set (usually at the plane):

We denote by $\mathcal{H}^1$ the linear Haurdorff measure (roughly speaking, length).

Problem (Euclidean Steiner problem)

Let C be a compact subset of $\mathbb{R}^d$. To find a closed S such that $S \cup C$ is connected and $\mathcal{H}^1(S)$ is minimal.

If C is totally disconnected then S should be connected.

Statement of Steiner problem

Problem: to connect a (usually finite) set of points by the shortest connected set (usually at the plane):

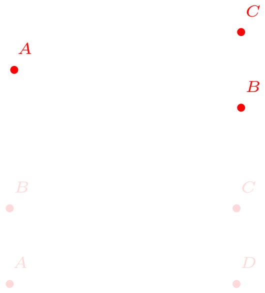
We denote by $\mathcal{H}^1$ the linear Haurdorff measure (roughly speaking, length).

Problem (Euclidean Steiner problem)

Let C be a compact subset of $\mathbb{R}^d$. To find a closed S such that $S \cup C$ is connected and $\mathcal{H}^1(S)$ is minimal.

If C is totally disconnected then S should be connected.

Steiner problem for three and four points. General properties



Torrichelli point is unique.

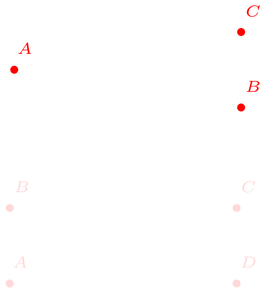
No uniqueness :(

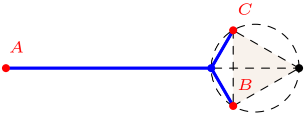
Problem (Euclidean Steiner problem)

Let C be a compact subset of $\mathbb{R}^d$. To find closed S such that $S \cup C$ is connected and $\mathcal{H}^1(S)$ is minimal.

- S exists;
- S contains no loops;
- only two variants of neighbourhoods for points from $S \setminus C$.

Steiner problem for three and four points. General properties





Torrichelli point is unique.

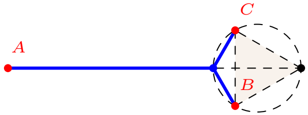
No uniqueness :(

Problem (Euclidean Steiner problem)

Let C be a compact subset of $\mathbb{R}^d$. To find closed S such that $S \cup C$ is connected and $\mathcal{H}^1(S)$ is minimal.

- S exists;
- S contains no loops;
- only two variants of neighbourhoods for points from $S \setminus C$.

Torrichelli point is unique.

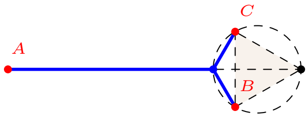


No uniqueness :(

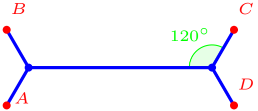
Problem (Euclidean Steiner problem)

Let C be a compact subset of $\mathbb{R}^d$. To find closed S such that $S \cup C$ is connected and $\mathcal{H}^1(S)$ is minimal.

- S exists;
- S contains no loops;
- only two variants of neighbourhoods for points from $S \setminus C$.



Torrichelli point is unique.



No uniqueness :(

Problem (Euclidean Steiner problem)

Let C be a compact subset of $\mathbb{R}^d$. To find closed S such that $S \cup C$ is connected and $\mathcal{H}^1(S)$ is minimal.

- S exists;
- S contains no loops;
- only two variants of neighbourhoods for points from $S \setminus C$.

Problem (Euclidean Steiner problem)

Let C be a compact subset of $\mathbb{R}^d$. To find closed S such that $S \cup C$ is connected and $\mathcal{H}^1(S)$ is minimal.

Some properties (in a full generality proved by Stepanov and Paolini, 2013)

- S exists;
- S contains no loops;
- Only two variants of a neighbourhood of a point $x \in S \setminus C$:
  - a regular tripod (x is a **branching point**);
  - a segment; x is an inner point.
- S contains at most countable number of branching points

Usually (if $C \subset S$) $S \cup C$ is called **Steiner tree**, and it is called **indecomposable** (full, irreducible), when $S \setminus C$ is connected. Three directions

Problem (Euclidean Steiner problem)

Let C be a compact subset of $\mathbb{R}^d$. To find closed S such that $S \cup C$ is connected and $\mathcal{H}^1(S)$ is minimal.

Some properties (in a full generality proved by Stepanov and Paolini, 2013)

- S exists;
- S contains no loops;
- Only two variants of a neighbourhood of a point $x \in S \setminus C$:
  - a regular tripod (x is a **branching point**);
  - a segment; x is an inner point.
- S contains at most countable number of branching points

Usually (if $C \subset S$) $S \cup C$ is called **Steiner tree**, and it is called **indecomposable** (full, irreducible), when $S \setminus C$ is connected. Three directions

Problem (Euclidean Steiner problem)

Let C be a compact subset of $\mathbb{R}^d$. To find closed S such that $S \cup C$ is connected and $\mathcal{H}^1(S)$ is minimal.

Some properties (in a full generality proved by Stepanov and Paolini, 2013)

- S exists;
- S contains no loops;
- Only two variants of a neighbourhood of a point $x \in S \setminus C$:
  - a regular tripod (x is a **branching point**);
  - a segment; x is an inner point.
- S contains at most countable number of branching points

Usually (if $C \subset S$) $S \cup C$ is called **Steiner tree**, and it is called **indecomposable** (full, irreducible), when $S \setminus C$ is connected. Three directions

Problem (Euclidean Steiner problem)

Let C be a compact subset of $\mathbb{R}^d$. To find closed S such that $S \cup C$ is connected and $\mathcal{H}^1(S)$ is minimal.

Some properties (in a full generality proved by Stepanov and Paolini, 2013)

- S exists;
- S contains no loops;
- Only two variants of a neighbourhood of a point $x \in S \setminus C$:
  - a regular tripod (x is a **branching point**);
  - a segment; x is an inner point.
- S contains at most countable number of branching points

Usually (if $C \subset S$) $S \cup C$ is called **Steiner tree**, and it is called **indecomposable** (full, irreducible), when $S \setminus C$ is connected. Three directions

Theorem (Paolini–Stepanov–T, 2015; Cherkashin–T. 2023; Paolini–Stepanov 2023)

There is a compact planar set M for which the unique solution of the Steiner problem has infinite number of triple points.

M, Σ are self-similar fractals with sufficiently small scale.

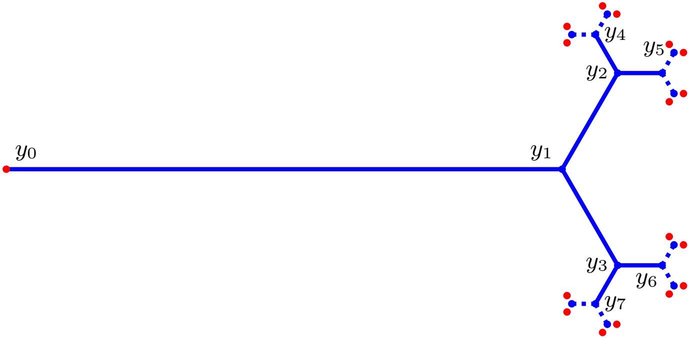


Figure: Indecomposable Steiner tree with infinite number of triple points

We say that full Steiner tree S connecting finite set $C = \{A_1 \dots A_n\}$ has *adding property* if there exists such $\varepsilon > 0$ that $\bigcup_{i=1}^{n}[A_iB_i] \cup S$ is Steiner tree for $\{B_1 \dots B_n\}$ where $|A_iB_i| = \varepsilon$ and B_i belongs to the ray beginning from the segment of S incident A_i.

Theorem

Cherkashin, T., 2022; reformulated Let St be a Steiner tree for terminals $A = (A_1, ..., A_m)$, $A_i \in \mathbb{R}^n$ such that every Steiner tree for an n-tuple in the closed $2r$-neighbourhood of A has the same topology as St for some positive r. Then St has adding property.

Example: the tripod

Usually the condition holds:

Theorem (Basok, Cherkashin, T., 2022)

For $m \geq 4$ the set of m terminals with non unique Steiner trees has the Hausdorff dimension $2m - 1$.

What if the condition does not hold?

We say that full Steiner tree S connecting finite set $C = \{A_1 \dots A_n\}$ has *adding property* if there exists such $\varepsilon > 0$ that $\bigcup_{i=1}^{n} [A_i B_i] \cup S$ is Steiner tree for $\{B_1 \dots B_n\}$ where $|A_i B_i| = \varepsilon$ and B_i belongs to the ray beginning from the segment of S incident A_i.

Theorem

Cherkashin, T., 2022; reformulated Let St be a Steiner tree for terminals $A = (A_1, ..., A_m)$, $A_i \in \mathbb{R}^n$ such that every Steiner tree for an n-tuple in the closed $2r$-neighbourhood of A has the same topology as St for some positive r. Then St has adding property.

Example: the tripod
Usually the condition holds:

Theorem (Basok, Cherkashin, T., 2022)

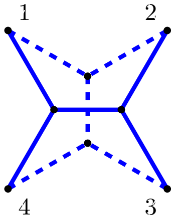
For $m \geq 4$ the set of m terminals with non unique Steiner trees has the Hausdorff dimension $2m - 1$.

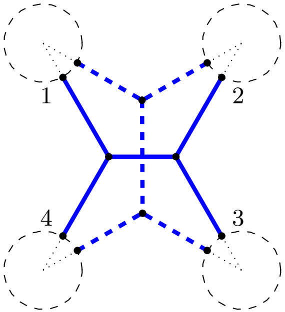
What if the condition does not hold?

Theorem (Cherkashin, T., 2022)

Let St be a Steiner tree for terminals $A = (A_1, ..., A_m)$, $A_i \in \mathbb{R}^n$ such that every Steiner tree for an n-tuple in the closed $2r$-neighbourhood of A has the same topology as St for some positive r. Then St has adding property.

It turns out that a Steiner tree for the vertices of a square does not have this property:





Thank you for your attention!

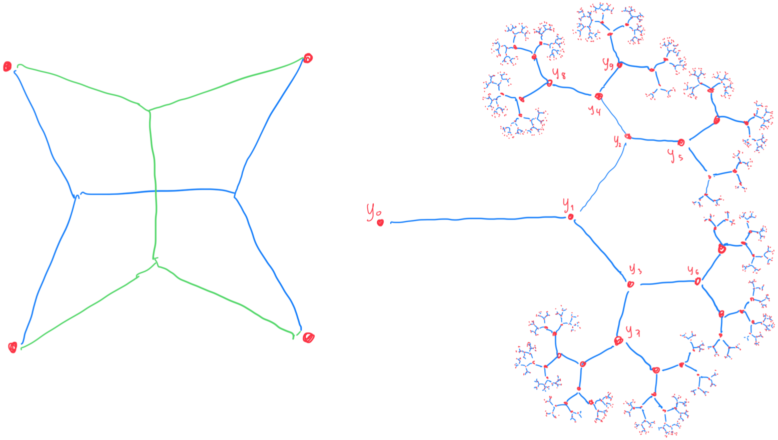


Figure: The left part contains two Steiner trees connecting vertices of a square; the right part provides an example of a Steiner tree with an infinite number of branching points y_i, $i \geq 1$.

Let (X, ρ) be a metric space. For any subset $U \subset X$, let $\operatorname{diam} U$ denote its diameter, that is $\operatorname{diam} U := \sup\{\rho(x, y) : x, y \in U\}$, $\operatorname{diam} \emptyset := 0$.
Let S be any subset of X, and $\delta > 0$ a real number. Define

$$H^d_\delta(S) = \inf\left\{\sum_{i=1}^{\infty} (\operatorname{diam} U_i)^d : \bigcup_{i=1}^{\infty} U_i \supseteq S, \operatorname{diam} U_i < \delta\right\}$$

where the infimum is over all countable covers of S by sets $U_i \subset X$ satisfying $\operatorname{diam} U_i < \delta$.
Let

$$H^d(S) := \sup_{\delta > 0} H^d_\delta = \lim_{\delta \to 0} H^d_\delta(S).$$