Solutions of N-body harmonic oscillators and Calogero-Moser model using Φ^4 matrix model

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The N-body harmonic oscillator system and its generalized system, the Calogero-Moser model, are known as quantum integrable systems, i.e. their Schrödinger equations are solvable and eigenstates of the Hamiltonians can be constructed. It has long been known that there is a connection between these quantum integrable systems and certain kind of generalized matrix models. This talk is concerned with the new correspondence of the quantum solvable systems with some matrix models, which is discovered last year. These matrix models are given as the Grosse-Wulkenhaar models, known as renormalizable scalar Φ^4-theories on Moyal spaces, which are non-commutative spaces. The Moyal space has Fock representation, so field theories can be expressed by using matrix representation. A scalar Φ^4-theory on Moyal space corresponds to a Hermitian Φ^4-matrix model or a real symmetric Φ^4-matrix model. In particular, Φ^4-matrix model known as the Grosse-Wulkenhaar models have kinetic terms $\mathrm{Tr}(E\Phi^2)$, where E is a positive diagonal matrix without degenerate eigenvalues. (These matrix models are also obtained by changing the potential of the Kontsevich model from Φ^3 to Φ^4.) We show that their partition functions of these matrix models correspond to zero-energy solutions of a Schrödinger type equation with the N-body harmonic oscillator Hamiltonian and the Calogero-Moser Hamiltonian, respectively.

Let Φ be a Hermitian or real symmetric $N \times N$ matrix. Let $Z(E, \eta)$ be the partition function defined by

$$Z(E,\eta) = \int d\Phi \; e^{-S_E[\Phi]}, \tag{1}$$

where $S_E = N \; \mathrm{Tr}\{E\Phi^2 + \frac{\eta}{4}\Phi^4\}$ and η is a real number. The domains of integrations are the space of Hermitian $N \times N$-matrices and the space of real symmetric $N \times N$-matrices, respectively. Let $\Delta(E)$ be the Vandermonde determinant $\Delta(E) := \prod_{k<l}(E_l - E_k)$. Using these, the theorem obtained can be described as follows.

Theorem 1. *Let $\Psi(E,\eta)$ be a function defined by*

$$\Psi(E,\eta) := e^{-\frac{N}{\beta\eta}\sum_{i=1}^N E_i^2}\Delta(E)^{\frac{\beta}{2}} Z(E,\eta).$$

Then $\Psi(E,\eta)$ is a zero-energy solution of the Schrödinger type equation

$$\mathcal{H}\Psi(E,\eta) = 0.$$

Here $\mathcal{H}$ is the Hamiltonian $\mathcal{H}_{HO}$ for the N-body harmonic oscillator system when we consider the Hermitian matrix model with $\beta = 2$:

$$\mathcal{H}_{HO} := -\frac{\eta}{N}\sum_{i=1}^N \left(\frac{\partial}{\partial E_i}\right)^2 + \frac{N}{\eta}\sum_{i=1}^N (E_i)^2.$$

When we consider the real symmetric matrix model with $\beta = 1$, then $\mathcal{H}$ is the Hamiltonian $\mathcal{H}_{CM}$ for Calogero-Moser model:

$$\mathcal{H}_{CM} := \frac{-\eta}{2N}\left(\sum_{i=1}^N \frac{\partial^2}{\partial E_i^2} + \frac{1}{4}\sum_{i\neq j}\frac{1}{(E_i - E_j)^2}\right) + 2\frac{N}{\eta}\sum_{i=1}^N E_i^2. \tag{2}$$

It is known that the N-body harmonic oscillator system or the Calogero-Moser model is associated with a Virasoro algebra structure. Using this fact, families of differential equations satisfied by the partition functions are also obtained from the Virasoro(Witt) algebra representations:

$$[\widetilde{L}_n, \widetilde{L}_m] = (n-m)\widetilde{L}_{n+m}. \tag{3}$$

The definitions of symbols and terms are left to the references [1, 2], but the following theorem is obtained

Theorem 2. *The partition function defined by (1) satisfies*

$$\mathcal{L}_{SD}(\widetilde{L}_{-m}Z(E,\eta)) = -\,2m(\widetilde{L}_{-m}Z(E,\eta)). \tag{4}$$

Here $\mathcal{L}_{SD}$ is a differential operator such that some Schwinger-Dyson equation for the partition function given by

$$\mathcal{L}_{SD}Z(E,\eta) = 0. \tag{5}$$

This means that $\widetilde{L}_{-m}Z(E,\eta)$ is an eigenfunction of $\mathcal{L}_{SD}$ with the eigenvalue $-2m$.

This talk is based on [1], in collaboration with H. Grosse, and [2], in collaboration with H. Grosse, N. Kanomata, and R. Wulkenhaar.

References

[1] H. Grosse and A. Sako, "Integrability of Φ^4 Matrix Model as N-body Harmonic Oscillator System," Lett Math Phys 114, 48 (2024). https://doi.org/10.1007/s11005-024-01783-2 [arXiv:2308.11523 [math-ph]].

[2] H. Grosse, N. Kanomata, A. Sako and R. Wulkenhaar, "Real symmetric Φ^4-matrix model as Calogero-Moser model," Lett Math Phys 114, 25 (2024). https://doi.org/10.1007/s11005-024-01772-5 [arXiv:2311.10974[hep-th]].