On the connection between algebraic, geometric, and topological methods in the classification of algebraic surfaces and curves

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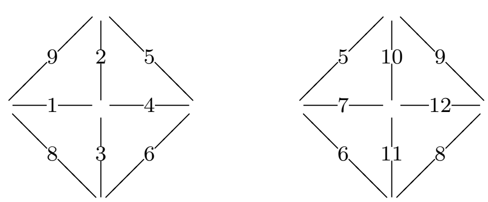
The classification of algebraic curves and surfaces in a moduli space is a challenging subject in algebraic geometry. Moduli spaces are spaces that parameterize families of algebraic surfaces. They can be used to study the geometry of algebraic surfaces and to compare different surfaces. Classifying algebraic surfaces and curves is an important task because of the comparison between different objects that we study. The moduli space of curves for example, is a space that parameterizes families of algebraic curves of a fixed genus.

The objects we study and classify can be algebraic curves (via fundamental groups and finding Zariski pairs), algebraic surfaces, and gluing of algebraic surfaces (via deformations and projections).

There are methods that can assist in this classification, for example: topological classification, intersection theory, singularities, cohomology, symmetric groups, etc. There are known algorithmic methods as well, and the choice of a method depends on the specific properties of the surface or curve in question and the desired level of detail in the classification.

From the geometric and topological point of view: we consider planar and non-planar deformations and projections to find branch curves of algebraic surfaces. From the algebraic and computational point of view: researchers in the mathematical community use the computer programs Magma, Singular, Maple, and so on. These are just a few examples of software packages that can be used for classifying algebraic surfaces and curves. In our research we use Magma as well, because we investigate fundamental groups and the Magma is a great tool for this goal. We have built some computer softwares to overcome the complicated algebraic computations in the fundamental groups.

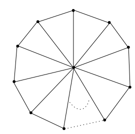
Firstly, in order to understand the complexity of the computations in the classification of algebraic surfaces, let us look at the following figure. We can see a high multiplicity of singularities. It happens especially when we glue two planar deformations or when we consider a non-planar deformation. The following figure shows two planar deformations glued together along four edges, and we get a non-planar deformation with multiplicity 4 in all singularities. In this case, the fundamental group of the Galois cover of the surface that has such a deformation is metabelian of order 2^{23} [1].



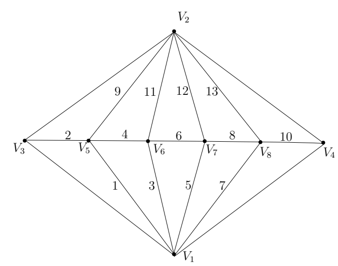
Now we explain how classification of algebraic surfaces works. We take an algebraic surface embedded in a projective space and project it with a generic projection onto the projective plane. We get the branch curve and then we are able to calculate G - the fundamental group of its complement. A special software gives as an output all braids relating to the branch curve and also the presentation of G. We then find a certain quotient of G, which is going to be the fundamental group of the Galois cover of the surface. This latter group is an invariant in the classification of algebraic surfaces, and has a geometric significance because it is equal for all the surfaces in the same connected component in the moduli space. We can define an isomorphism between our group and some Coxeter quotient, and we can determine the fundamental groups, using the ideas in [2].

Details about the fundamental groups of Galois cover of an algebraic surface and some interesting examples can be found in our works [3, 4, 5, 6]. In these recent works, we study algebraic surfaces with deformations that have Zappatic R_n singularities (for any n) [3], and also their gluings [4], and surfaces that have non-planar deformations, in which singularities with high complexity appear [6]. Moreover, we study deformations with Zappatic E_n singularities [5].

Theorem 1 ([5]). *The fundamental group of the Galois cover of surfaces that have deformation with one Zappatic E_n singularity is trivial for $n \geq 4$.*



Theorem 2. *Galois covers of a union of two Zappatic surfaces of type R_n are simply-connected surfaces of general type, for any n.*



As for algebraic curves, we can use the software that we constructed in order to produce braids and presentations of fundamental groups and determine these fundamental groups. These computations enable us to get Zariski pairs, see examples in [7, 8]. Moreover, the study of families of curves is an inseparable part of the classification of curves because there we can also calculate invariants, perform deformations, and check how these processes affect the classification [9].

References

[1] Meirav Amram, Cheng Gong, Praveen Kumar Roy, Uriel Sinichkin, Uzi Vishne. The fundamental group of Galois covers of surfaces with octahedral envelope. https://doi.org/10.48550/arXiv.2303.05241

[2] Meirav Amram, Ruth Lawrence, Uzi Vishne. Artin covers of the braid groups. *Journal of Knot Theory and its Ramifications*, **21**(7), 44pp, 2012. https://doi.org/10.1142/S0218216512500617

[3] Meirav Amram, Cheng Gong, JiaLi Mo. On the Galois covers of deformations of surfaces of minimal degree. *Mathematische Nachrichten*, 296(40) : 1351-1365, 2023. https://doi.org/10.1002/mana.202100183

[4] Meirav Amram, Cheng Gong, JiaLi Mo. On Galois covers of a union of Zappatic surfaces of type R_k. https://arxiv.org/abs/2401.06170

[5] Meirav Amram, Cheng Gong, JiaLi Mo. Deformations of Zappatic stable surfaces and their Galois covers. https://arxiv.org/abs/2402.06017

[6] Meirav Amram. Non-planar degenerations and related fundamental groups. https://arxiv.org/abs/2104.02781

[7] Meirav Amram, Robert Shwartz, Uriel Sinichkin, ShengLi Tan, Hiro-O Tokunaga. Zariski pairs of conic-line arrangements of degrees 7 and 8 via fundamental groups. https://arxiv.org/abs/2106.03507.ry

[8] Meirav Amram, Shinzo Bannai, Uriel Sinichkin, Taketo Shirane, Hiro-O Tokunaga. The realization space of a certain conic line arrangement of degree 7 and a π_1-equivalent Zariski pair. https://arxiv.org/abs/2307.01736

[9] Meirav Amram, Praveen Kumar Roy, Uriel Sinichkin. Fundamental groups of highly symmetrical curves and Fermat line arrangments. https://arxiv.org/abs/2310.04365