Nilpotent structures of oriented neutral vector bundles

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Let E be an oriented vector bundle over a manifold M of rank $4n$ and h a neutral metric of E. We call a section N of $\mathrm{End}\,E$ a *nilpotent structure* of E if on a neighborhood of each point of M, there exists an ordered frame field $e = (e_1, \dots, e_{2n}, e_{2n+1}, \dots, e_{4n})$ of E satisfying

$$h(e_i, e_i) = -h(e_{2n+i}, e_{2n+i}) = 1 \ (i = 1, \dots, 2n), \quad h(e_i, e_j) = 0 \ (i \neq j) \tag{1}$$

and $Ne = e\Lambda_n$, where

$$\Lambda_n := \begin{bmatrix} O_n & -I_n & O_n & I_n \\ I_n & O_n & I_n & O_n \\ O_n & I_n & O_n & -I_n \\ I_n & O_n & I_n & O_n \end{bmatrix},$$

I_n is the $n \times n$ unit matrix and O_n is the $n \times n$ zero matrix. Let N be a nilpotent structure of E. We call N an *ε-nilpotent structure* ($\varepsilon \in \{+, -\}$) if on a neighborhood of each point of M, there exists an ordered frame field e giving the orientation of E and satisfying (1) and $NeI'_{4n,\varepsilon} = eI'_{4n,\varepsilon}\Lambda_n$ with

$$I'_{4n,\varepsilon} := \begin{bmatrix} I_n & O_n & O_n & O_n \\ O_n & I_n & O_n & O_n \\ O_n & O_n & I_n & O_n \\ O_n & O_n & O_n & I_{n,\varepsilon} \end{bmatrix}, \qquad I_{1,\pm} := \pm 1, \quad I_{n,\pm} := \begin{bmatrix} \pm 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix} \ (n \geq 2).$$

Let N be an ε-nilpotent structure of E. Then such a frame field as e is called an *admissible frame field* of N. For an admissible frame field e of N, we set $\xi = \xi_1 \wedge \cdots \wedge \xi_{2n}$, where

$$\begin{aligned} \xi_1 &:= e_1 - e_{2n+1}, & \xi_i &:= e_i - e_{2n+i}, \\ \xi_{n+1} &:= e_{n+1} + \varepsilon e_{3n+1}, & \xi_{n+i} &:= e_{n+i} + e_{3n+i} \end{aligned} \quad (i = 2, \dots, n).$$

Then ξ does not depend on the choice of an admissible frame field e of N ([3]). Therefore N gives a section ξ_N of the $2n$-fold exterior power $\bigwedge^{2n} E$ of E. A nilpotent structure is characterized by

(i) $\mathrm{Im}\,N = \mathrm{Ker}\,N$, and $\pi_N := \mathrm{Im}\,N = \mathrm{Ker}\,N$ is a light-like subbundle of E of rank $2n$,
(ii) $h(\phi, N\phi) = 0$ for any local section ϕ of E

([2], [3]). In particular, N gives a null structure on each fiber of E and h is null-Hermitian with respect to N (see [9]). The subbundle π_N is locally spanned by $\xi_1, \dots, \xi_{2n}$.

Remark Suppose $n = 1$. Then $\bigwedge^2 E$ is a vector bundle over M of rank 6 and h induces a metric $\hat{h}$ of $\bigwedge^2 E$ of signature (2,4). In addition, $\bigwedge^2 E$ is decomposed as $\bigwedge^2 E = \bigwedge^2_+ E \oplus \bigwedge^2_- E$ by two subbundles $\bigwedge^2_+ E$, $\bigwedge^2_- E$ of rank 3 and the restriction of $\hat{h}$ on each of them has signature (1,2). The *light-like twistor spaces* associated with E are fiber bundles $U_0\left(\bigwedge^2_\pm E\right)$ in $\bigwedge^2_\pm E$ respectively such that each fiber is a light cone. Each light-like line subbundle of $\bigwedge^2_+ E$ or $\bigwedge^2_- E$ corresponds to a light-like subbundle of E of rank 2 and each ε-nilpotent structure N of E corresponds to a section of $U_0\left(\bigwedge^2_\varepsilon E\right)$ given by $(1/\sqrt{2})\xi_N$ ([2], [3]). The space-like twistor spaces $U_+\left(\bigwedge^2_\pm E\right)$ associated with E are fiber bundles in $\bigwedge^2_\pm E$ respectively such that each fiber is a hyperboloid of two sheets. A section of $U_+\left(\bigwedge^2_\varepsilon E\right)$

corresponds to a complex structure of E preserving h. See [1], [5] for the space-like twistor spaces. The time-like twistor spaces $U_-\left(\bigwedge^2_\pm E\right)$ associated with E are fiber bundles in $\bigwedge^2_\pm E$ respectively such that each fiber is a hyperboloid of one sheet. A section of $U_-\left(\bigwedge^2_\varepsilon E\right)$ corresponds to a paracomplex structure of E reversing h. See [1], [13], [14] for the time-like twistor spaces. See [7], [10], [11] for the twistor spaces in the case h is a Riemannian (i.e., positive-definite) metric, which are the prototypes of $U_+\left(\bigwedge^2_\pm E\right)$, $U_-\left(\bigwedge^2_\pm E\right)$ and $U_0\left(\bigwedge^2_\pm E\right)$.

Let ∇ be a connection of E satisfying $\nabla h = 0$. Let N be an ε-nilpotent structure of E. We say that N satisfies the *Walker condition* with respect to ∇ if for any local section ψ of π_N, $\nabla\psi$ is a 1-form valued in π_N. See [6], [9], [16] for Walker manifolds. Let $\hat{\nabla}$ be the connection of $\bigwedge^{2n} E$ induced by ∇. Then N satisfies the Walker condition with respect to ∇ if and only if $\hat{\nabla}\xi_N = \alpha \otimes \xi_N$ for a 1-form α. If $\nabla N = 0$, then $\hat{\nabla}\xi_N = 0$ ([4]) and therefore N satisfies the Walker condition ([9]).

The main objects of research in this talk are special nilpotent structures, and they are called *H-nilpotent structures* of (E, h, ∇), where H is a Lie subgroup of $SO(2n, 2n)$ related to neutral hyperKähler structures. There exist a complex structure I and paracomplex structures J_1, J_2 of E such that h, ∇, I, J_1, J_2 form a neutral hyperKähler structure of E if and only if there exists an H-nilpotent structure of (E, h, ∇) ([4]). See [5], [12] for paraquaternionic structures. See [8], [15] for neutral hyperKähler 4-manifolds.

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References

[1] N. Ando, Surfaces with zero mean curvature vector in neutral 4-manifolds, Diff. Geom. Appl. **72** (2020) 101647, 31 pp.
[2] N. Ando, Nilpotent structures of neutral 4-manifolds and light-like surfaces, Developments in Lorentzian Geometry, Springer Proceedings in Mathematics & Statistics **389**, Springer, 2022, 13–28.
[3] N. Ando, Sections of time-like twistor spaces with light-like or zero covariant derivatives, preprint; arXiv:2305.14741.
[4] N. Ando, Nilpotent structures of oriented neutral vector bundles, preprint; arXiv:2405.05003.
[5] D. Blair, J. Davidov and O. Muškarov, Hyperbolic twistor spaces, Rocky Mountain J. Math. **35** (2005) 1437–1465.
[6] M. Brozos-Vázquez, E. García-Río, P. Gilkey, S. Nikčević, R. Vázquez-Lorenzo, The geometry of Walker manifolds, Synthesis Lectures on Mathematics and Statistics, Morgan and Claypool, 2009.
[7] R. Bryant, Conformal and minimal immersions of compact surfaces into the 4-sphere, J. Differential Geom. **17** (1982) 455–473.
[8] J. Davidov, G. Grantcharov, O. Muškarov and M. Yotov, Compact complex surfaces with geometric structures related to split quaternions, Nuclear Physics B **865** (2012) 330–352.
[9] M. Dunajski, Null Kähler geometry and isomonodromic deformations, Commun. Math. Phys. **391** (2022) 77–105.
[10] J. Eells and S. Salamon, Twistorial construction of harmonic maps of surfaces into four-manifolds, Annali della Scuola Normale Superiore di Pisa, Classe di Scienze **12** (1985) 589–640.
[11] T. Friedrich, On surfaces in four-spaces, Ann. Glob. Anal. Geom. **2** (1984) 257–287.
[12] E. García-Río, Y. Matsushita, and R. Vázquez-Lorenzo, Paraquaternionic Kähler manifolds, Rocky Mountain J. Math. **31** (2001) 237–260.
[13] K. Hasegawa and K. Miura, Extremal Lorentzian surfaces with null τ-planar geodesics in space forms, Tohoku Math. J. **67** (2015) 611–634.
[14] G. Jensen and M. Rigoli, Neutral surfaces in neutral four-spaces, Matematiche (Catania) **45** (1990) 407–443.
[15] H. Kamada, Neutral hyperKähler structures on primary Kodaira surfaces, Tsukuba J. Math. **23** (1999) 321–332.
[16] A. G. Walker, Canonical form for a Riemannian space with a parallel field of null planes, Quart. J. Math. Oxford (2) **1** (1950) 69–79.