Algebra in fields extended by infinity

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Definition 1. A *corps* is a set F endowed with two binary operations $+,\cdot : F\times F\to F$ and two distinct constants $0,1\in F$ that satisfy the following eight axioms:

(1) $\forall x,y,z\in F\ \big(x+(y+z)=(x+y)+z\big)$;
(2) $\forall x\in F\ \ (x+0=x=0+x)$;
(3) $\forall x\in F\ \exists y\in F\ (x+y=0=y+x)$;
(4) $\forall x,y,z\in F\ \ (x\cdot(y\cdot z)=(x\cdot y)\cdot z)$;
(5) $\forall x\in F\ (x\cdot 1=x=1\cdot x)$;
(6) $\forall x\in F\setminus\{0\}\ \exists y\in F\ \ (x\cdot y=1=y\cdot x)$;
(7) $\forall a,x,y\in F\ \ \big(a\cdot(x+y)=a\cdot x+a\cdot y\big)$;
(8) $\forall x,y,b\in F\ \ \big((x+y)\cdot b=x\cdot b+y\cdot b\big)$.

A corps F is called a *field* if $x\cdot y=y\cdot x$ for all elements $x,y\in F$.

Definition 2. A *procorps*[1] is a set F endowed with two binary operations $+,\cdot : F\times F\to F$ and three distinct constants $0,1,\infty\in F$ that satisfy the following nine axioms:

(1) $\forall x,z\in F\ \forall y\in F\setminus\{\infty\}\ \ \big(x+(y+z)=(x+y)+z\big)$;
(2) $\forall x,y\in F\ \ (x+y=y+x)$;
(3) $\forall x\in F\ \ (x+0=x=0+x)$;
(4) $\forall x,z\in F\ \forall y\in F\setminus\{0,\infty\}\ \ (x\cdot(y\cdot z)=(x\cdot y)\cdot z)$;
(5) $\forall x\in F\ (x\cdot 1=x=1\cdot x)$;
(6) $\forall x\in F\ \exists y\in F\ \ (x\cdot y=1=y\cdot x)$;
(7) $\forall a\in F\setminus\{0,\infty\}\ \forall x,y\in F\ \ \big(a\cdot(x+y)=a\cdot x+a\cdot y\big)$;
(8) $\forall x,y\in F\ \forall b\in F\setminus\{0,\infty\}\ \ \big((x+y)\cdot b=x\cdot b+y\cdot b\big)$;
(9) $0\cdot 0=0$, $\infty\cdot\infty=\infty$ and $1+\infty=\infty=\infty+1$.

A procorp F is called a *profield* if $x\cdot y=y\cdot x$ for all elements $x,y\in F$.

Example 3. Let F be a corps and $\infty\notin F$. Consider the set $\bar F:=F\cup\{\infty\}$, and extend the operations of addition and multiplication from F to $\bar F$ letting

$$\begin{aligned}&\forall x\in\bar F\setminus\{\infty\}\ \ (x+\infty=\infty=\infty+x),\\&\forall x\in\bar F\setminus\{0\}\ \ (x\cdot\infty=\infty=\infty\cdot x),\\&\infty+\infty=0,\quad \infty\cdot 0=1=0\cdot\infty.\end{aligned}$$

The set $\bar F$ endowed with the extended operations of addition and multiplication and the constants $0,1,\infty$ is a procorps, called the *projective ∞-extension* of the corps F. If F is a field, then its projective ∞-extension $\bar F$ is a profield.

Example 4. The Riemannian sphere $\bar{\mathbb C}=\mathbb C\cup\{\infty\}$ is the projective ∞-extension of the field of complex numbers $\mathbb C$.

The following theorem shows that procorps are exactly projective ∞-extensions of corps.

Theorem 5. *For every procorps (profield) $\bar F$, the set $F:=\bar F\setminus\{\infty\}$ endowed with the induced operations of addition and multiplication is a corps (field) and $\bar F$ is the projective ∞-extension of F.*

[1] *Procorps* is an abbreviation of a “projective corps”.