Geometric and algebraic-topological structures in Schwartz distribution spaces for relativistic Quantum Mechanics

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The classic Hilbert space methods cannot be used for the definition and resolution of the free relativistic Schrödinger equation because the very fundamental solutions of this equation cannot be framed in a Hilbert or Banach space context. We can justify the necessity to use distribution theory, for many reasons. If we lock down ourselves in separable Hilbert space theory, we cannot hope to solve satisfactorily (from a physical point of view) the relativistic Schrödinger equation for free particles. The simple reason is that the very main (and generating) solutions of the relativistic Schrödinger equation for free particles cannot be considered as elements of a separable Hilbert Space. First of all, if we desire to consider the standard L^2 product, we immediately observe that the de Broglie waves do not belong to the space of square integrable functions. Moreover, the set of all harmonic waves is a continuous set (not a discrete one) and, if we select its "naturally orthonormal" subfamily (that generating the unitary Minkowsky-Fourier transform as integral kernel), we are getting again a continuous family that should be orthogonal by right, from a physical perspective, but cannot be as such in any separable Hilbert space (even different from L^2). We cannot find continuous orthonormal families in a separable Hilbert space, but only discrete orthonormal systems! Furthermore, even forcing the matter and considering a Hilbert space generated by all those "unitary orthogonal waves", we would obtain a non-separable Hilbert space, that would complicate enormously the matter, from a functional calculus point of view, because we have no reasonable or natural spectral theory for non-separable Hilbert spaces. We are not saying, here, that we have not to use nonseparable Hilbert spaces in Quantum mechanics, but we see that they do not help in the formulation and resolution of the relativistic Schrödinger equation.

Analogous problems we would risk to face if we lock down ourselves in separable Normed Space theory: it is very hard to keep, in a unique functional theory, a reasonable and convenient separable norm with a good spectral theory and the presence of the continuous family of de Broglie waves. That is a general problem, in quantum mechanics and quantum field theory: when we consider harmonic waves and related differential equations (or operatorial equations), theoretical phisicists, actually, do not use Hilbert Space theory - and we know it - we use, instead, smooth function theory, differentiable function theory, we work, essentially, with calculus techniques and distribution theory. Then, we have another general problem of Hilbert space theory in QM: even when we solve the classic Dirac free equation, the manipulations and resolutions proceed, essentially, in a differentiable theory context; indeed, the basis solutions of the free Dirac equation are bispinors constructed by harmonic waves and then, automatically, we work out of the Hilbert space theory. Moreover, when we work out of the Hilbert space theory, we work also out of the spectral theory on Hilbert spaces. Consequently, we cannot expect to find a correct and unambiguous definition of the square root of an operator by Hilbert space techniques, if the domain of such operator should contain the de broglie waves, because in this case we are playing outside of any separable Hilbert space. It's not a case (and do not surprise at all) that Dirac equation was solved by smooth calculus and finite algebraic methods, rather than infinite dimensional Hilbert space techniques. In order to define the square-roots of differential operators, in quantum mechanical context (where we need to manage harmonic waves, eigenstates of position operator, continuous spectra, and so on...), we need a spectral theory constructed elsewhere, not in Hilbert spaces. In some way, quantum mechanics needs to coordinate and put together two apparently incompatible aspects: the state space of a quantum system can be generated (simultaneously) by discrete and continuous bases, at the same time: the position and momentum basis ($|x>$), ($|p>$)

are continuous, while the Hermite function basis is discrete. Very often we read "let's solve the harmonic oscillator problem in the position basis", or "let's solve the harmonic oscillator problem in the momentum basis", which are continuous basis (in some Radon-Pettis-Schwartz sense to be correctly defined) only to see, after a while, that the harmonic oscillator is solved by the discrete Hermite function basis of L^2 (it would be better to say of the Schwartz function Space $\mathcal{S}$). How the position basis and momentum basis (that completely stay out of L^2) can generate the same state space generated by the L^2 Hermite function basis? In what sense a continuous family of vectors can generate a functional space? The position eigenstates are not even functions, they are measures. In what space are we moving? The state space is separable or not? What is its Hilbert dimension, $\aleph_0$ or $\aleph_1$? How a separable Hilbert (or Banach) space could contain "non-normalizable" vectors and continuous orthogonal families of non-normalizable vectors that, from a physical point of view, represent simply the certainty to observe a specific result? In tempered distribution spaces, we know that the Hermite function family is a discrete basis in a rightful algebraic-topological sense, it is a total family, it is also a basis in a generalized Hilbert sense (with respect to the tempered distribution topology). Moreover, position and momentum basis lives in $\mathcal{S}'$ and generate $\mathcal{S}'$ in the Schwartz linear algebraic sense S' is a separable topological vector space, it is wonderfully generated by discrete and continuous basis, in two different rigorous and operative meanings and, by the way, exactly the meaning used practically by quantum physicists, in a more heuristic way. In addition to that, in the distribution approach, any quantum mechanics observable is a continuous and everywhere defined operator, while in the Hilbert space approach we almost surely face discontinuous (unbounded) operators and very strange, unnatural domains, even for the most simple observables (position, momentum, ladder operators, number operator and so on and so forth). Consequently, in Hilbert spaces, we face any kind of difficulties, even to add or multiply two straightforward operators such as a derivative operator and the position operator - which show different domains - that without, and well before, coming to ask "what the principal square root of a discontinuous, non-everywhere defined, not-properly hermitian, densely defined (or perhaps closable) operator is". What our new Schwartz linear Algebra theory clarifies compared to previous methods? The new theory clarifies definitely where we are working, in quantum theories and quantum field theory, especially in relativistic quantum mechanics. We clarify that the state spaces of quantum objects are not Hilbert spaces (if we exclude the finite dimensional spaces that, anyway, are subspaces of the tempered distribution space $\mathcal{S}'$), but powers of tempered distribution spaces. Fortunately, tempered distribution spaces contain a lot of good inner product spaces, useful in the evaluation of probabilities and expectation values for quantum mechanics. By those inner products, we finally understand the possible way to "normalize" properly the eigenvectors of the position and momentum operators (eigenstates that should be normalizable because of their straightforward physical meaning: they simply represent certainty). First of all, we solve the problem of evolution in the tempered distribution space. Then, when necessary, we go to find the right inner product subspace, of tempered distribution space, in which calculate probabilities and expectation values.

References

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