Lie subalgebras of real order-three special linear Lie algebra revisited

Yevhenii Chapovskyi
(Institute of Mathematics of NAS of Ukraine, Kyiv, Ukraine)
E-mail: e.chapovskyi@imath.kiev.ua

Serhii D. Koval
(Memorial University of Newfoundland, Canada
and Institute of Mathematics of NAS of Ukraine, Kyiv, Ukraine)
E-mail: koval.srh@imath.kiev.ua

Olha Zhur
(Faculty of Mechanics and Mathematics,
National Taras Shevchenko University of Kyiv, Kyiv, Ukraine)
E-mail: oliazhur@knu.ua

The study of subalgebras within both real and complex Lie algebras presents a complex challenge, which arises in many fields of mathematics and its applications. For instance, listing inequivalent subalgebras of the maximal Lie invariance algebra of a system of differential equations could result in constructing its "inequivalent" exact solutions. These classifications also serve as an efficient tool in the realms of theoretical physics and the study of integrable systems. At the same time, they themselves remain to be interesting algebraic problems.

In [1] we review the entire framework of the subalgebra classification problem following [2] and references therein, and also suggest new points of view on these methods and rigorously present their theoretical framework. We apply the developed enhanced methods for refining the classification of subalgebras of the Lie algebra $\mathfrak{sl}_3(\mathbb{R})$ and as a byproduct we first obtain the complete classification of the subalgebras of real rank-two affine Lie algebra $\mathfrak{aff}_2(\mathbb{R})$. The real order-three special linear Lie algebra $\mathfrak{sl}_3(\mathbb{R})$ is the algebra of traceless 3×3 matrices with the standard matrix commutator as the Lie bracket and it is spanned by the matrices

$$E_1:=\begin{pmatrix}0&0&0\\1&0&0\\0&0&0\end{pmatrix},\quad E_2:=\frac12\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix},\quad E_3:=\begin{pmatrix}0&-1&0\\0&0&0\\0&0&0\end{pmatrix},\quad D:=\frac16\begin{pmatrix}1&0&0\\0&1&0\\0&0&-2\end{pmatrix},$$

$$P_1=\begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix},\quad P_2=\begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix},\quad R_1=\begin{pmatrix}0&0&0\\0&0&0\\0&-1&0\end{pmatrix},\quad R_2=\begin{pmatrix}0&0&0\\0&0&0\\1&0&0\end{pmatrix}.$$

In this way, the algebra $\mathfrak{sl}_3(\mathbb{R})$ is defined through its faithful irreducible representation of the minimal dimension, which is exactly the vector space $\mathbb{R}^3$.

The best attempt in listing inequivalent subalgebras of $\mathfrak{sl}_3(\mathbb{R})$ was carried out in [2], however it contains a number of misprints, mistakes and the major result is presented without proof. The analogous subalgebra classification for the Lie algebras $\mathfrak{sl}_n(\mathbb{R})$, $n\geqslant 4$, remains to be a significant open problem.

To classify Lie algebras of a simple Lie algebra $\mathfrak{sl}_3(\mathbb{R})$ modulo $\mathrm{SL}_3(\mathbb{R})$-equivalence we adopt the approach detailed in [1, Section 2.1]. Specifically for $\mathfrak{sl}_3(\mathbb{R})$ we go through the following steps:

- (*i*) using the defining representation $\mathbb{R}^3$ of the Lie algebra $\mathfrak{sl}_3(\mathbb{R})$ determine all its maximal reducibly and irreducibly embedded subalgebras;
- (*ii*) for each of the identified maximal subalgebras construct the lists of inequivalent subalgebras with respect to the action of their corresponding inner automorphism groups;
- (*iii*) merge the obtained lists modulo the action of the group $\mathrm{SL}_3(\mathbb{R})$.

This analysis reveal that the Lie algebra $\mathfrak{sl}_3(\mathbb{R})$ contains two irreducibly embedded maximal subalgebras: the special orthogonal Lie algebras $\mathfrak{so}_3(\mathbb{R})$ and $\mathfrak{so}_{2,1}(\mathbb{R})$, as well as two reducibly embedded maximal subalgebras $\mathfrak{a}_1 = \langle E_1, E_2, E_3, E_4, D, P_1, P_2\rangle$ and $\mathfrak{a}_2 = \langle E_1, E_2, E_3, E_4, D, R_1, R_2\rangle$. Both of the latter subalgebras are isomorphic to the rank-two affine Lie algebra $\mathfrak{aff}_2(\mathbb{R})$. According to the step (ii), the classification of the subalgebras of $\mathfrak{aff}_2(\mathbb{R})$ is an essential step in the course of addressing the primary problem.

The Lie algebra $\mathfrak{aff}_2(\mathbb{R})$ is the semidirect product $\mathfrak{gl}_2(\mathbb{R}) \ltimes \mathbb{R}^2$. Therefore, to classify subalgebras of $\mathfrak{aff}_2(\mathbb{R})$ we apply the approach for classifying Lie subalgebras of the semidirect products from [1, Section 2.3] to the Lie algebra $\mathfrak{gl}_2(\mathbb{R}) \ltimes \mathbb{R}^2$. Consequently, we present for the first time the complete list of inequivalent subalgebras of the rank-two affine Lie algebra $\mathfrak{aff}_2(\mathbb{R})$ in [1, Theorem 11]. In fact, the classification of subalgebras of the algebra $\mathfrak{aff}_2(\mathbb{R})$ was initiated in [2, Section 3.3], where its inequivalent "twisted" and "nontwisted" subalgebras were listed, however this classification were not completed. Moreover, the validity of these lists is questionable, since to construct them it is essential to have the correct classification of subalgebras of $\mathfrak{gl}_2(\mathbb{R})$, which in [2, eq. (3.11)] was presented with a mistake and a number of misprints. This was an additional motivation for us to thoroughly and comprehensively classify the subalgebras of $\mathfrak{aff}_2(\mathbb{R})$.

To combine the derived lists of inequivalent subalgebras $\mathfrak{a}_1$ and $\mathfrak{a}_2$ modulo the $\mathrm{SL}_3(\mathbb{R})$-equivalence, we specify the following general proposition to the case of the Lie algebra $\mathfrak{sl}_3(\mathbb{R})$.

Proposition 1. *Let $\mathfrak{m} \subset \mathfrak{g}$ be a Lie subalgebra, $M \subset G$ the corresponding Lie subgroup. Choose a subset $C \subset G$ such that $MC = G$. Then Lie subalgebras $\mathfrak{h}_1 \subset \mathfrak{m}$ and $\mathfrak{h}_2 \subset \mathfrak{g}$ are conjugate if and only if there exists $g \in C$ such that $\mathrm{Ad}_g\mathfrak{h}_2 \subset \mathfrak{m}$ and moreover $\mathrm{Ad}_g\mathfrak{h}_2 \sim \mathfrak{h}_1$ up to $\mathrm{Inn}(\mathfrak{m})$-equivalence.*

Theorem 2. *A complete list of proper $\mathrm{SL}_3(\mathbb{R})$-inequivalent subalgebras of the algebra $\mathfrak{sl}_3(\mathbb{R})$ is exhausted by the subalgebras, where $\varepsilon \in \{-1, 1\}$, $\delta \in \{0, 1\}$, $\kappa \geqslant 0$, $\mu \in [-1, 3]$, $\mu' \in [0, 1]$ and $\gamma \in \mathbb{R}$:*

1D: $\mathfrak{f}^{\delta}_{1.1} = \langle E_1 + \delta P_1\rangle$, $\mathfrak{f}^{\gamma}_{1.2} = \langle E_1 + E_3 + \gamma D\rangle$, $\mathfrak{f}^{\mu'}_{1.3} = \langle E_2 + \mu' D\rangle$, $\mathfrak{f}_{1.4} = \langle E_1 + D\rangle$,

2D: $\mathfrak{f}_{2.1} = \langle P_1, P_2\rangle$, $\mathfrak{f}^{\delta}_{2.2} = \langle E_1 + \delta P_1, P_2\rangle$, $\mathfrak{f}_{2.3} = \langle E_2 + D + P_2, P_1\rangle$, $\mathfrak{f}_{2.4} = \langle E_1 + D, P_2\rangle$,
$\mathfrak{f}_{2.5} = \langle E_1, D\rangle, \mathfrak{f}_{2.6} = \langle E_2, D\rangle, \mathfrak{f}_{2.7} = \langle E_1 + E_3, D\rangle$,
$\mathfrak{f}^{\gamma}_{2.8} = \langle E_2 + \gamma D, E_1\rangle, \mathfrak{f}_{2.9} = \langle E_2 - 3D, E_1 + P_1\rangle$,

3D: $\mathfrak{f}_{3.1} = \langle E_1, P_1, P_2\rangle$, $\mathfrak{f}_{3.2} = \langle D, P_1, P_2\rangle$, $\mathfrak{f}^{\kappa}_{3.3} = \langle E_2 + \kappa D, P_1, P_2\rangle$,
$\mathfrak{f}^{\varepsilon}_{3.4} = \langle E_1 + \varepsilon D, P_1, P_2\rangle$, $\mathfrak{f}^{\gamma}_{3.5} = \langle E_1 + E_3 + \gamma D, P_1, P_2\rangle$, $\mathfrak{f}^{\gamma}_{3.6} = \langle E_1 + E_3 + \gamma D, R_1, R_2\rangle$,
$\mathfrak{f}^{\mu}_{3.7} = \langle E_2 + \mu D, E_1, P_2\rangle$, $\mathfrak{f}_{3.8} = \langle E_2 - 3D, E_1 + P_1, P_2\rangle$, $\mathfrak{f}_{3.9} = \langle E_1, E_2, D\rangle$,
$\mathfrak{f}_{3.10} = \langle E_1, E_2, E_3\rangle, \mathfrak{f}_{3.11} = \langle E_1 + E_3, P_1 - R_2, P_2 + R_1\rangle$,
$\mathfrak{f}_{3.12} = \langle E_1 + E_3, P_1 + R_2, P_2 - R_1\rangle$,

4D: $\mathfrak{f}_{4.1} = \langle E_1, D, P_1, P_2\rangle$, $\mathfrak{f}_{4.2} = \langle E_2, D, P_1, P_2\rangle$, $\mathfrak{f}_{4.3} = \langle E_1 + E_3, D, P_1, P_2\rangle$,
$\mathfrak{f}^{\gamma}_{4.4} = \langle E_2 + \gamma D, E_1, P_1, P_2\rangle$, $\mathfrak{f}_{4.5} = \langle E_1, E_2, D, P_2\rangle$, $\mathfrak{f}_{4.6} = \langle E_1 + E_3, D, R_1, R_2\rangle$,
$\mathfrak{f}_{4.7} = \langle E_1, E_2, E_3, D\rangle$,

5D: $\mathfrak{f}_{5.1} = \langle E_1, E_2, D, P_1, P_2\rangle$, $\mathfrak{f}_{5.2} = \langle E_1, E_2, E_3, P_1, P_2\rangle$, $\mathfrak{f}_{5.3} = \langle E_1, E_2, E_3, R_1, R_2\rangle$,

6D: $\mathfrak{f}_{6.1} = \langle E_1, E_2, E_3, D, P_1, P_2\rangle$, $\mathfrak{f}_{6.2} = \langle E_1, E_2, E_3, D, R_1, R_2\rangle$.

References

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[2] Winternitz, P., Subalgebras of Lie algebras. Example of $\mathfrak{sl}(3, \mathbb{R})$, in *Symmetry in physics*, *CRM Proc. Lecture Notes*, Vol. 34, Amer. Math. Soc., Providence, RI, 2004, pp. 215–227.