Conformal mappings and a non-holonomic frame

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It is convenient to use a holonomic coordinate systems and associated the so-called natural frame if one explore conformal mappings of differentiable manifolds. But to study some physical applications, in particular spinor fields, we have to use non-holonomic frame which sometimes referred to as the vielbein formulation [1, 2] . In this formulation we introduce 4 independent vectors $t^i_a(x)$, $(a = 0, 1, 2, 3)$ at each point of a spacetime $(V^{1,3}, g)$, which are orthogonal to each other and have a unit length:

$$t^i_a(x)t^j_b(x)g_{ij}(x) = \eta_{ab}, \quad \eta_{ab} = diag(1, -1, -1, -1).$$

Also there exists the inverse matrix t^a_i, which satisfies

$$t^i_a(x)t^a_j(x) = \delta^i_j, \quad t^i_a(x)t^b_i(x) = \delta^a_b.$$

The field $t^a_i(x)$ is called the vielbein [1]. Using such non-holonomic coordinate systems, we should introduce the spin connection by the relation below

$$\omega_k{}^a{}_b = (t^i_b\Gamma^h_{ki} + \partial_k t^h_b)t^a_h. \tag{1}$$

From (1) it follows that

$$\partial_k t^h_a + \Gamma^h_{jk}t^j_a - \omega_k{}^b{}_a t^h_b = 0.$$

The covariant derivative of a spinor field $\psi(x)$ one calculates using the formula:

$$\nabla_k\psi = \partial_k\psi - \frac{1}{4}\omega_{kab}\gamma^{ab}\psi = \partial_k\psi + \Gamma_k\psi,$$

where $\gamma^{ab} = \frac{1}{2}(\gamma^a\gamma^b - \gamma^b\gamma^a)$ is the antisymmetrized product of two gamma matrices. For the adjoint spinor $\overline{\psi} = \psi^\dagger\gamma^0$ we have

$$\nabla_k\overline{\psi} = \partial_k\overline{\psi} + \overline{\psi}\frac{1}{4}\omega_{kab}\gamma^{ab} = \partial_k\overline{\psi} - \overline{\psi}\Gamma_k,$$

If we consider a conformal mapping $f : (V^{1,3}, g) \to (\tilde{V}^{1,3}, \tilde{g})$, i.e. $\tilde{g}_{ij} = e^{2\varphi(x)}g_{ij}$, then the vielbein is transformed as

$$\tilde{t}^a_i(x) = e^{\varphi(x)}t^a_i(x). \tag{2}$$

Here $\varphi(x)$ is some function. Under a conformal transformation the spin connection transforms as

$$\tilde{\omega}_{kab} = \omega_{kab} + t_{ka}\varphi_b - t_{kb}\varphi_a.$$

Here $\varphi_b = \partial_b\varphi = t^j_b\partial_j\varphi$. Hence for the spin-affine connection Γ_k we get:

$$\tilde{\Gamma}_k = \Gamma_k - \frac{1}{4}(t_{ka}\varphi_b - t_{kb}\varphi_a)\gamma^{ab} = \Gamma_k - \frac{1}{2}t_{ka}\varphi_b\gamma^{ab}. \tag{3}$$

On the other hand, the stress-energy tensor for the spinor field ($s = \frac{1}{2}$) in a spacetime $(V^{1,3}, g)$ we could calculate by the formula [3]:

$$T_{jk} = \frac{i}{2}\big(\overline{\psi}\gamma_{(j}\nabla_{k)}\psi - (\nabla_{(j}\overline{\psi})\gamma_{k)}\psi\big), \tag{4}$$

where $\gamma_j = \gamma_a t^a_j(x)$. Taking into account (2), (3), (4) we obtain the transformed stress-energy tensor:

$$\tilde{T}_{jk} = e^{\varphi(x)}\big(T_{jk} - \frac{i}{4}\big(\overline{\psi}\gamma_j t_{ka}\varphi_b\gamma^{ab}\psi + \overline{\psi}\gamma_k t_{ja}\varphi_b\gamma^{ab} + \overline{\psi}t_{ka}\varphi_b\gamma^{ab}\gamma_j + \overline{\psi}t_{ja}\varphi_b\gamma^{ab}\gamma_k\psi\big)\big),$$

However we have the scalar which is preserved under conformal mappings:

$$|A|^2 = A^i g_{ij} A^j = \overline{\psi}\gamma^i\psi g_{ij}\overline{\psi}\gamma^j\psi,$$

where $A^i = \overline{\psi}\gamma^i\psi$ is so called four-dimensional current of the spinor field ψ.

References

[1] P. Collas, D. Klein, *The Dirac Equation in Curved Spacetime*, SpringerBriefs in Physics, 2019.
[2] T. Yoshiaki, *Introduction to Supergravity*, Springer Tokyo Heidelberg New York Dordrecht London, 2018.
[3] N. D. Birrell, P. C. W. Davies,*Quantum Fields in Curved Space*, Cambridge University Press, England, 1982.