On topologization of the bicyclic monoid

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In this paper we shall follow the terminology of [2, 4, 5, 6].

A semigroup S is called *inverse* if for any element $x \in S$ there exists a unique $x^{-1} \in S$ such that $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$. The element x^{-1} is called the *inverse of* $x \in S$. If S is an inverse semigroup, then the function $\operatorname{inv}\colon S \to S$ which assigns to every element x of S its inverse element x^{-1} is called the *inversion*. On an inverse semigroup S the semigroup operation determines the *natural partial order* $\preccurlyeq$ on S: $s \preccurlyeq t$ if and only if there exists $e \in E(S)$ such that $s = te$.

A topology τ on a semigroup S is called:

- a *semigroup* (*shift-continuous*) topology if (S, τ) is a topological (semitopological) semigroup;
- an *inverse semigroup* topology if (S, τ) is a topological inverse semigroup;
- an *inverse shift-continuous* topology if (S, τ) is a semitopological semigroup with continuous inversion.

The bicyclic monoid $\mathcal{C}(p, q)$ is the semigroup with the identity 1 generated by two elements p and q subjected only to the condition $pq = 1$. The bicyclic monoid admits only the discrete semigroup Hausdorff topology [3]. Bertman and West in [1] extended this result for the case of Hausdorff semitopological semigroups.

We construct two non-discrete inverse semigroup T_1-topologies and a compact inverse shift-continuous T_1-topology on the bicyclic monoid $\mathcal{C}(p, q)$. Also we give conditions on a T_1-topology τ on $\mathcal{C}(p, q)$ to be discrete.

Theorem 1. *Every shift-continuous Baire T_1-topology τ on the bicyclic monoid $\mathcal{C}(p, q)$ is discrete.*

Theorem 2. *Let τ be an inverse semigroup T_1-topology on $\mathcal{C}(p, q)$. If there exists a point $q^i p^j \in \mathcal{C}(p, q)$ such that the space $\updownarrow_{\preccurlyeq} q^i p^j$ is quasi-regular at $q^i p^j$, then τ is discrete.*

Theorem 3. *Let τ be a shift-continuous T_1-topology on the bicyclic monoid $\mathcal{C}(p, q)$ such that the maps $\mathcal{C}(p, q) \to E(\mathcal{C}(p, q))$, $x \mapsto xx^{-1}$ and $\mathcal{C}(p, q) \to E(\mathcal{C}(p, q))$, $x \mapsto x^{-1}x$ are continuous. If there exists a point $q^i p^j \in \mathcal{C}(p, q)$ such that the space $\updownarrow_{\preccurlyeq} q^i p^j$ is semiregular at $q^i p^j$, then τ is discrete.*

References

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