An Application to Sasaki Extremal metrics via the Berglund-Hübsch rule

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Sasaki-extremal metrics were introduced in [1] as a generalization of metrics with constant scalar curvature, which is obstructed by the Futaki invariant. On this talk we exhibit examples of homotopy spheres and rational homology spheres realized as links of chain-cycle polynomials that do not admit Sasaki extremal metrics in the whole Sasaki cone, which has dimension greater than one. For this, we consider links that are given as 2-fold branched covers of S^9 whose branching loci are rational homology 7-spheres which are links of certain invertible polynomials of chain-cycle type studied in [5] and later in [4] through the Berglund-Hübsch rule of classical mirror symmetry. In [2], Boyer and van Coevering defined a relative version of the K-stability of Collins and Székelyhidy [6, 7] and obtain the first examples of Sasaki manifolds with Sasaki cone of dimension greater than one not admitting Sasaki extremal metrics in the whole Sasaki cone. Based on their result we exhibit 37 new families of links with Sasaki cone of dimension 2 such the whole cone does not admit any extremal representative. All the examples produced here are eitheir homotopy 9-spheres, rational homology 9-spheres or manifolds of the form $S^4 \times S^5$. We can easily extrapolate these examples to arbitrary dimensions where the corresponding Sasaki cones have larger dimensions. All these examples are inequivalent to the ones found in [2]. These examples are consequences of the following more general result that we present in this talk:

Proposition 1 ([5]). *Consider a polynomial of chain-cycle type of the form*

$$f = z_0^{a_0} + z_0 z_1^{a_1} + z_4 z_2^{a_2} + z_2 z_3^{a_3} + z_3 z_4^{a_4}$$

with $a_1 = 2$ whose link L_f is a rational homology sphere and that cuts out a projective hypersurface in $\mathbb{P}(w_0, w_1, \ldots, w_4)$ such that

$$(w_0, w_1, w_2, w_3, w_4) = (m_3 v_0, m_3 v_1, m_2 v_2, m_2 v_3, m_2 v_4),$$

with m_3 odd, $\gcd(m_2, m_3) = 1$ and $d = m_3 m_2$. The polynomial

$$g = f^T + z_5^2 + \ldots + z_n^2$$

with f^T the Berglund-Hübsch transpose of f, determines a link L_g such that

(1) *If n is even, then L_g is a rational homology $(2n-1)$-sphere.*
(2) *If n is odd, then L_g is a homotopy $(2n-1)$-sphere and $\Delta_g(-1) = m_3$. In particular the diffeomorphism type of L_g is determined by m_3.*
(3) *The Sasaki cone of L_g has dimension $1 + \lfloor \frac{n-3}{2} \rfloor$ and there are no extremal Sasaki metrics in the Sasaki cone of the link L_g.*
(4) *If m_3 is even, then for $n = 5$, the link L_g has the form $S^4 \times S^5$.*

References

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