On singularities of mappings with a Lebesgue integrable majorant

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The following definitions are from [1]. A path γ in ${\Bbb R}^n$ is a continuous mapping $\gamma:\Delta\rightarrow{\Bbb R}^n$ where Δ is an interval in ${\Bbb R}$. Its locus $\gamma(\Delta)$ is denoted by $|\gamma|$. Given a family Γ of paths γ in ${\Bbb R}^n$, a Borel function $\rho:{\Bbb R}^n\rightarrow [0,\infty]$ is called *admissible* for Γ, abbr. $\rho\in{\rm adm}\,\Gamma$, if

$$\int\limits_{\gamma}\rho(x)|dx|\geqslant 1$$

for each (locally rectifiable) $\gamma\in\Gamma$. Given $p\geqslant 1$, the *p-modulus* of Γ is defined by the relation

$$M_p(\Gamma):=\inf\limits_{\rho\in{\rm adm}\,\Gamma}\int\limits_{{\Bbb R}^n}\rho^p(x)dm(x) \qquad (1)$$

interpreted as $+\infty$ if ${\rm adm}\,\Gamma=\varnothing$.

Given sets E and F and a given domain D in $\overline{{\Bbb R}^n}={\Bbb R}^n\cup\{\infty\}$, we denote by $\Gamma(E,F,D)$ the family of all paths $\gamma:[0,1]\rightarrow \overline{{\Bbb R}^n}$ joining E and F in D, that is, $\gamma(0)\in E$, $\gamma(1)\in F$ and $\gamma(t)\in D$ for all $t\in(0,1)$. Everywhere below, unless otherwise stated, the boundary and the closure of a set are understood in the sense of the extended Euclidean space $\overline{{\Bbb R}^n}$. Let $x_0\in\overline{D}$, $x_0\ne\infty$,

$$S(x_0,r)=\{x\in{\Bbb R}^n: |x-x_0|=r\}\,, S_i=S(x_0, r_i)\,,\quad i=1,2\,,$$
$$A=A(x_0, r_1, r_2)=\{x\in{\Bbb R}^n: r_1<|x-x_0|<r_2\}\,.$$

Let $f:D\rightarrow{\Bbb R}^n$, $n\geqslant 2$, and let $Q:{\Bbb R}^n\rightarrow [0,\infty]$ be a Lebesgue measurable function such that $Q(y)\equiv 0$ for $y\in{\Bbb R}^n\setminus f(D)$. Let $A=A(y_0, r_1, r_2)$ and $\Gamma_f(y_0, r_1, r_2)$ denotes the family of all paths $\gamma:[a, b]\rightarrow D$ such that $f(\gamma)\in \Gamma(S(y_0, r_1), S(y_0, r_2), A(y_0, r_1, r_2))$, i.e., $f(\gamma(a))\in S(y_0, r_1)$, $f(\gamma(b))\in S(y_0, r_2)$, and $f(\gamma(t))\in A(y_0, r_1, r_2)$ for any $a<t<b$. We say that f *satisfies the inverse Poletsky inequality at $y_0\in f(D)$ with respect to p-modulus,* if the relation

$$M_p(\Gamma_f(y_0, r_1, r_2))\leqslant \int\limits_{A} Q(y)\cdot \eta^p(|y-y_0|)\, dm(y) \qquad (2)$$

holds for any $0<r_1<r_2< r_0:=\sup\limits_{y\in f(D)}|y-y_0|$ and any Lebesgue measurable function $\eta:(r_1, r_2)\rightarrow [0,\infty]$ such that

$$\int\limits_{r_1}^{r_2}\eta(r)\, dr\geqslant 1\,. \qquad (3)$$

Note that estimates of the type (2) are well known and hold at least for $p=n$ in many classes of mappings (see, e.g., [2, Theorem 3.2], [3, Theorem 6.7.II] and [4, Theorem 8.5]). For $p\ne n$, similar estimates may be found, e.g., in [5] and [6].

A mapping $f:D\rightarrow {\Bbb R}^n$ is called *discrete* if the image $\{f^{-1}\left(y\right)\}$ of any point $y\,\in\,{\Bbb R}^n$ consists of isolated points, and *open* if the image of any open set $U\subset D$ is an open set in ${\Bbb R}^n$.

Later, in the extended space $\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$ we use the *spherical (chordal) metric* $h(x,y) = |\pi(x) - \pi(y)|$, where π is a stereographic projection of $\overline{\mathbb{R}^n}$ onto the sphere $S^n(\frac{1}{2}e_{n+1}, \frac{1}{2})$ in $\mathbb{R}^{n+1}$, namely,

$$h(x,\infty) = \frac{1}{\sqrt{1+|x|^2}}\,,$$

$$h(x,y) = \frac{|x-y|}{\sqrt{1+|x|^2}\sqrt{1+|y|^2}}\,, \quad x \neq \infty \neq y \tag{4}$$

(see, e.g., [1, Definition 12.1]). The following statement is true.

Theorem 1. *Let $n \geqslant 2$, $p \geqslant n$, let D be a domain in $\mathbb{R}^n$, $x_0 \in D$, and let $f : D \setminus \{x_0\} \to \mathbb{R}^n$ be an open discrete mapping that satisfies the conditions* (2)-(3) *at any point $y_0 \in \overline{D'} \setminus \{\infty\}$, where $D' := f(D \setminus \{x_0\})$.*

If $Q \in L^1(D')$, then f has a continuous extension $\overline{f} : D \to \overline{\mathbb{R}^n}$, the continuity of which should be understood in the sense of the chordal metric h in (**??**). *The extended mapping $\overline{f}$ is open and discrete in D. Moreover, if $p = n$ and $\overline{f}(x_0) \neq \infty$, then there is a neighborhood $U \subset D$ of the point x_0 depending only on x_0, and $C = C(n, D, x_0) > 0$ such that*

$$|\overline{f}(x) - \overline{f}(x_0)| \leqslant \frac{C_n \cdot (\|Q\|_1)^{1/n}}{\log^{1/n}\left(1 + \frac{\delta}{2|x-x_0|}\right)} \tag{5}$$

for any $x, y \in U$, where $\|Q\|_1$ is the norm of the function Q in $L^1(D')$.

References

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