Inequalities involving means in high-dimensional spaces

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The objects of study are convex bodies: compact, convex subsets of Euclidean spaces. Convexity naturally appears in many areas of mathematics, such as Linear Programming, Probability Theory, Functional Analysis, Partial Differential Equations, Information Theory and Geometry of Numbers.

For instance, density functions of some of the most important probability measures are logarithmically (or at least quasi) concave functions, like gaussians, exponential, or uniform densities over convex domains. In particular, this means that all their level sets are convex. Although convexity is a simple to formulate property, convex bodies possess a surprisingly rich structure. The main subject of the proposed project are geometric inequalities and extreme relations between convex sets in general. Especially, we are interested in extending results given so far only for symmetric convex sets or join results given separately for the symmetric case and the general case. To do so we want to take a functional into account that measures how far a convex body C is away from being symmetric. One such functional is the so called *Minkowski (measure of) asymmetry*, which measures in terms of the Banach-Mazur distance how far a set is from its closest symmetric set.

We start explaining some notation, which is mostly standard in convex geometry. For details see, e.g. [10]. For any $A, B \subset \mathbb{R}^n$ let $A \subset_t B$ denote that there exists a translate of A being a subset of B. The *Minkowski addition* of two sets $A, B \subset \mathbb{R}^n$ is given by the set $A + B = \{x + y : x \in A, y \in B\}$. Moreover, for any n-dimensional convex set K we denote by $\rho K := \{\rho x : x \in K\}$ and $-K := (-1)K$. For any set convex set K, we say that K is *symmetric* if $K =_t -K$. Moreover, let $K^\circ = \{x \in \mathbb{R}^n : x^\perp y \leq 1 \, \forall y \in K\}$ be the *polar* of K.

The main object of study in this project is the *Minkowski asymmetry* of a convex set C, defined as

$$s(C) = \min\{\rho > 0 : C \subset_t -\rho C\},$$

where we are allowed to write min instead of inf as C is a convex set, and this is true for all similar functionals we define below. Moreover, if $c - C \subset s(C)(c - C)$ we say that c is a *Minkowski center* of C, and if $c = 0$, we say that C is *Minkowski centered*. It is well known (see e.g. [8]) that for all convex sets C we have $s(C) \in [1, n]$ with $s(C) = 1$ if and only if C is symmetric and $s(C) = n$ if and only if C is an *n-simplex*, i.e., the convex hull of $n + 1$ affinely independent points.

Naturally, the Minkowski sum of two convex sets defines a mean. The harmonic, geometric, and arithmetic means of real numbers a and b are collectively known as the Pythagorean means. They are related by the extended arithmetic-geometric-harmonic mean inequality (see [10]). Thus, for convex sets K and C the *arithmetic mean* is defined as $\frac{K+C}{2}$, while the *harmonic mean* is defined as $\left(\frac{K^\circ+C^\circ}{2}\right)^\circ$. The *minimum* and *maximum* of K and C are represented by $K \cap C$ and $\mathrm{conv}(K \cup C)$, respectively. In the 1960s, Firey introduced and studied different means of convex sets, known as p-means (see [6, 7]). This line of investigation continues to this day (see [9]).

Notice that the considered symmetrizations of a convex body K, i.e., $K \cap (-K)$, $\frac{K-K}{2}$, $\mathrm{conv}(K \cup (-K))$, are frequently used in convex geometry, e.g., as extreme cases of a variety of geometric inequalities. Consider, e.g., the Bohnenblust inequality [1], which bounds from above the ratio of the circumradius ($\min_{x \in \mathbb{R}^n} \max_{y \in K} \|x - y\|$) and the diameter ($\max_{x,y \in K} \|x - y\|$) of convex bodies in arbitrary normed spaces endowed with a norm $\|\cdot\|$ by $n/(n + 1)$, and for which equality is reached in spaces with $S \cap (-S)$ or $\frac{1}{2}(S - S)$ as the unit ball [5] where S is a 0-centered regular simplex. These means also appear in characterizations of spaces, for which K is complete or reduced [4, Prop. 3.5 – 3.10].

In [6] it was shown that similarly to the Pythagorean means, the means of convex sets can be ordered in terms of inclusions [6]. Thus, for any convex sets K, C with 0 in their interior we have

$$K \cap C \subset \left(\frac{K^\circ + C^\circ}{2}\right)^\circ \subset \frac{K+C}{2} \subset \operatorname{conv}(K \cup C). \tag{1}$$

For a Minkowski-centered convex compact set K we define the factor $\alpha(K)$ to be the smallest possible factor to cover $K \cap (-K)$ by $\operatorname{conv}(K \cup (-K))$, i.e.,

$$\alpha(K) := \inf\{\rho > 0 : K \cap (-K) \subset \rho \operatorname{conv}(K \cup (-K))\}.$$

In [2] we show a surprising result, showing that in 2-space the greatest value of the Minkowski asymmetry such that the harmonic mean can be optimally contained in the arithmetic mean is the golden ratio $\varphi = \frac{1+\sqrt{5}}{2} \approx 1.61$. Moreover, if $s(K) = \varphi$, there exists a non-singular linear transformation L, such that

$$L(K) = \operatorname{conv}\left(\left\{\begin{pmatrix}-1\\-1\end{pmatrix}, \begin{pmatrix}-1\\0\end{pmatrix}, \begin{pmatrix}0\\\varphi\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\-1\end{pmatrix}\right\}\right)$$

is the *golden house*.

We also present a family of planar sets K_s with $s(K_s) = s \in [1, \varphi]$, such that $\alpha(K) = 1$, thus, showing that for any $s \in [1, 2]$ there exists a planar Minkowski centered K with $s(K) = s$, $\alpha(K) = 1$.

In [3] we give a complete description the region of all possible values for $\alpha(K)$ for planar Minkowski centered K in dependence of the asymmetry of K, showing that

$$\frac{2}{s(K)+1} \le \alpha(K) \le \min\left\{1, \frac{s(K)}{s(K)^2 - 1}\right\}.$$

Moreover, for every pair (α, s), such that $\frac{2}{s+1} \le \alpha \le \min\left\{1, \frac{s}{s^2-1}\right\}$, there exists a Minkowski centered planar convex set K, such that $s(K) = s$ and $\alpha(K) = \alpha$.

Surprisingly, in the same paper we were able to describe the number of intersection points of the boundaries of a convex set K and its negative $-K$, when its asymmetry is greater than the golden ratio. Namely, we show that for any Minkowski centered K with $s(K) \ge \varphi$ the set $\operatorname{bd}(K) \cap \operatorname{bd}(-K)$ consists of exactly 6 points. However, when the asymmetry is less than the golden ratio, $\operatorname{bd}(K) \cap \operatorname{bd}(-K)$ can consist of countable or uncountable number of points, as well as of a small one.

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