Two problems in the theory of metric preserving functions

Viktoriia Bilet
(Institute of Applied Mathematics and Mechanics of the NASU, Cherkasy, Ukraine)
E-mail: viktoriiabilet@gmail.com

Oleksiy Dovgoshey
(Institute of Applied Mathematics and Mechanics of the NASU, Cherkasy, Ukraine;
University of Turku, Turun yliopisto, Finland)
E-mail: oleksiy.dovgoshey@gmail.com

The following is a particular case of J. Jachymski and F. Turoboś concept, see [1] for more details.

Definition 1. Let $\mathbf{A}$ be a class of metric spaces. Let us denote by $\mathbf{P_A}$ the set of all functions $f : [0,\infty) \to [0,\infty)$ such that the implication

$$((X, d) \in \mathbf{A}) \Rightarrow ((X, f \circ d) \in \mathbf{A})$$

is valid for every metric space (X, d).

We will use the following notations:

- $\mathbf{F}$, set of functions $f : [0,\infty) \to [0,\infty)$;
- $\mathbf{M}$, class of metric spaces;
- $\mathbf{U}$, class of ultrametric spaces;

Definition 2. A function $f \in \mathbf{F}$ is *metric preserving* (*ultrametric preserving*) iff $f \in \mathbf{P_M}$ ($f \in \mathbf{P_U}$).

Remark 3. The concept of metric preserving functions can be traced back to Wilson [2]. Similar problems were considered by Blumenthal in [3]. The theory of metric preserving functions was developed by Borsík, Doboš, Piotrowski, Vallin and other mathematicians. See also lectures by Doboš [4], and the introductory paper by Corazza [5]. The study of ultrametric preserving functions begun by P. Pongsriiam and I. Termwuttipong in 2014 [6].

Our main purpose is to give the answers on the following problems.

Problem 4. Let $\mathbf{A} \subseteq \mathbf{P_M}$. Find conditions under which the equation

$$\mathbf{P_X} = \mathbf{A} \tag{1}$$

has a solution $\mathbf{X} \subseteq \mathbf{M}$.

Problem 5. Let $\mathbf{A} \subseteq \mathbf{P_U}$. Find conditions under which equation (1) has a solution $\mathbf{X} \subseteq \mathbf{U}$.

Let us recall some basic concepts of semigroup theory, see, for example, John M. Howie [7].

A *semigroup* is a pair $(S, *)$ consisting of a nonempty set S and an associative operation $* : S \times S \to S$ which is called the *multiplication* on S. A semigroup $S = (S, *)$ is a *monoid* if there is $e \in S$ such that

$$e * s = s * e = s$$

for every $s \in S$.

Definition 6. Let $(S, *)$ be a semigroup and $\varnothing \neq T \subseteq S$. Then T is a *subsemigroup* of S if $a, b \in T \Rightarrow a * b \in T$. If $(S, *)$ is a monoid with the identity e, then T is a *submonoid* of S if T is a subsemigroup of S and $e \in T$.

Solutions to Problems 4 and 5 are given, respectively, in Theorems 7 and 8 below.

Theorem 7. *Let* $\mathbf{A}$ *be a nonempty subset of the set* $\mathbf{P_M}$ *of all metric preserving functions. Then the following statements are equivalent.*

(i) *The equality*

$$\mathbf{P_X} = \mathbf{A} \tag{2}$$

has a solution $\mathbf{X} \subseteq \mathbf{M}$.

(ii) $\mathbf{A}$ *is a submonoid of* $(\mathbf{F}, \circ)$.

(iii) $\mathbf{A}$ *is a submonoid of* $(\mathbf{P_M}, \circ)$.

The next theorem is an ultrametric analog of the previous theorem.

Theorem 8. *Let* $\mathbf{A}$ *be a nonempty subset of the set* $\mathbf{P_U}$ *of all ultrametric preserving functions. Then the following statements are equivalent.*

(i) *The equality* $\mathbf{P_X} = \mathbf{A}$ *has a solution* $\mathbf{X} \subseteq \mathbf{U}$.

(ii) $\mathbf{A}$ *is a submonoid of* $(\mathbf{F}, \circ)$.

(iii) $\mathbf{A}$ *is a submonoid of* $(\mathbf{P_U}, \circ)$.

Some properties of the monoids of $\mathbf{P_M}$ and $\mathbf{P_U}$ were described in [8] and [9].

Funding. Viktoriia Bilet was partially supported by a grant from the Simons Foundation (Award 1160640, Presidential Discretionary-Ukraine Support Grants, Viktoriia Bilet). Oleksiy Dovgoshey was supported by a grant of Turku University, Finland.

References

[1] J. Jachymski, F. Turoboś. On functions preserving regular semimetrics and quasimetrics satisfying the relaxed polygonal inequality. *Rev. R. Acad. Cienc. Exactas Fís. Nat.*, Ser. A Mat., RACSAM 114 (3): 159, 2020.

[2] W. A. Wilson. On certain types of continuous transformations of metric spaces. *American Journal of Mathematics* 57 (1): 62–68, 1935.

[3] L. Blumenthal. Remarks concerning the euclidean four-point property. *Ergeb. Math. Kolloq.* Wien 7: 7–10, 1936.

[4] J. Doboš. *Metric Preserving Functions.* Štroffek, Košice, Slovakia, 1998.

[5] P. Corazza. Introduction to metric-preserving functions. *Amer. Math. Monthly* 104 (4): 309–323, 1999.

[6] P. Pongsriiam, I. Termwuttipong. Remarks on ultrametrics and metric-preserving functions. *Abstr. Appl. Anal.* 1–9, 2014.

[7] J. M. Howie. *Fundamentals of semigroup theory.* Clarendan Press, Oxford, 1995.

[8] O. Dovgoshey, O. Martio. Functions trasferring metrics to metrics. *Beitrage zur Algebra und Geometrie.* 54(1): 237–261, 2013.

[9] O. Dovgoshey. Strongly ultrametric preserving functions. *Topology Appl.* https:// doi.org/10.1016/jtopol.2024.108931, 2024.