Group action on noncommutative curves

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Recall that a *noncommutative curve* (*noc*) is a pair $\mathbb{X} = (X, \mathcal{O}_{\mathbb{X}})$, where X is an algebraic curve (the *base curve* of $\mathbb{X}$) over a filed $\Bbbk$ and $\mathcal{O}_{\mathbb{X}}$ is a sheaf of $\mathcal{O}_X$-algebras coherent as a sheaf of $\mathcal{O}_X$-modules. We always suppose that $\mathcal{O}_X \subseteq \mathcal{O}_{\mathbb{X}}$ and the curve $\mathbb{X}$ is *reduced*, i.e. $\mathcal{O}_{\mathbb{X}}$ has no nilpotent ideals. We also suppose that $\Bbbk$ is algebraically closed.

The noc $\mathbb{X}$ is called *hereditary* if $\operatorname{gl.dim} \mathcal{O}_{\mathbb{X}} = 1$, that is $\mathcal{E}xt^n_{\mathcal{O}_{\mathbb{X}}}(\mathcal{M}, \mathcal{N}) = 0$ for $n > 1$ and all coherent sheaves of $\mathcal{O}_{\mathbb{X}}$-modules $\mathcal{M}, \mathcal{N}$. We denote by $\mathbb{X}^\sharp$ the noc $(\mathcal{O}_X, \mathcal{O}_{\mathbb{X}^\sharp})$, where for every point $x \in X$

$$\mathcal{O}_{\mathbb{X}^\sharp,x} = \begin{cases} \mathcal{O}_{\mathbb{X},x} & \text{if } \mathcal{O}_{\mathbb{X},x} \text{is hereditary,} \\ \mathcal{E}nd_{\mathbb{X},x}\, \mathfrak{r}_x & \text{otherwsise,} \end{cases}$$

where $\mathfrak{r}_x = \operatorname{rad} \mathcal{O}_{\mathbb{X},x}$. It is known that if $\mathfrak{r}_x = \operatorname{rad} \mathcal{O}_{\mathbb{X}^\sharp,x}$, the noc $\mathbb{X}$ is hereditary. In this case $\mathbb{X}$ is called a *Bäckström curve*. If, moreover, $\ell_{\mathcal{O}_{\mathbb{X}}}\big(\mathcal{O}_{\mathbb{X}^\sharp} \otimes_{\mathcal{O}_{\mathbb{X}}} U \leq 2$ for every simple sheaf of $\mathcal{O}_{\mathbb{X}}$-modules U, $\mathbb{X}$ is called a *nodal curve*. Note that a "usual" (commutative) curve X is nodal if and only if all its singularities are *simple nodes*, that is, if $x \in X$ is a singular point, $\hat{\mathcal{O}}_x \simeq \Bbbk[[x, y]]/(xy)$. The structure of nodal nocs is described in [1].

Let a finite group G acts on a noc $\mathbb{X}$. It means that it acts on the base curve X as well as on the sheaf of algebras $\mathcal{O}_{\mathbb{X}}$ (maybe with a factor set in the sense of [2]). The the noc $\mathbb{X} * G = (X/G, \mathcal{O}_{\mathbb{X}*G})$ (the *crossed product* of $\mathbb{X}$ and G) is defined. Note that G naturally acts on the category $\operatorname{Coh} \mathbb{X}$ of coherent $\mathcal{O}_{\mathbb{X}}$-modules.

Theorem 1. *Suppose that the order of the group G is invertible in $\Bbbk$.*

(1) There is an equivalence of categories $\operatorname{Coh}(\mathbb{X} * G)$ *and* $\operatorname{add}((\operatorname{Coh} \mathbb{X}) * G)$*, where* $\operatorname{add} \mathcal{C}$ *denotes the* Karubian closure *of the category $\mathcal{C}$, i.e. the smallest additive category containing $\mathcal{C}$ and such that all idempotents in it split.*

(2) If $\mathbb{X}$ is hereditary (Bäckström, nodal), so is $\mathbb{X} * G$.

References

[1] Igor Burban and Yuriy Drozd. *Non-commutative nodal curves and derived tame algebras.* arXiv:1805.05174 [math.AG] (2018)

[2] Natan Jacobson. *Theory of Rings.* AMS, 1943.