$6D$-Riemannian metric associated at the Navier-Stokes equations and its applications

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Theorem 1. *The $6D$ metric in local coordinates (t,x,y,z,v,w)*

$$ {d_s}^2 = -2\,B\left(t,x,y,z\right)d_t d_v + 2\,E\left(t,x,y,z\right)d_t d_w + d_t d_x + 2\,H\left(t,x,y,z\right)d_v d_w + d_v d_y - $$
$$ -2\int \frac{\partial}{\partial y} H\left(t,x,y,z\right)\,{\rm d}z {d_w}^2 + d_w d_z, $$

with conditions on coefficients

$$ \frac{\partial}{\partial y} H\left(t,x,y,z\right) - \frac{\partial}{\partial x} E\left(t,x,y,z\right) = 0,\ \ \frac{\partial}{\partial z} H\left(t,x,y,z\right) - \frac{\partial}{\partial x} B\left(t,x,y,z\right) = 0, $$
$$ \frac{\partial}{\partial z} E\left(t,x,y,z,t\right) - \frac{\partial}{\partial y} B\left(t,x,y,z\right) = 0 $$

is associated with equations of compatibility of the Navier-Stokes system o equations

$$ \vec{\nabla} P(\vec{t},x) = \frac{\partial}{\partial t}\vec{V} + \left(\vec{V}\cdot\vec{\nabla}\right)\vec{V} = \mu\Delta\vec{V}, \qquad \vec{\nabla}\cdot\vec{V} = 0 $$

with respect to the function of pressure in flow of liquid $P(\vec{t},x) = P(t,x,y,z)$. [1],[2]

On base of this theorem the examples of exact solutions of Navier-Stokes system of equations may be constructed.

The Ricci-tensor R_{ik} of given metric has six components $R_{tt}, R_{vv}, R_{vw}, R_{ww}, R_{tv}, R_{tw}$ and from conditions of compatibility between the various equations for R_{ik} lead determined the components of velocities $\vec{U}(t,\vec{x})$ and the function pressure $P(t,\vec{x})$.

As example simplest reduction of considered metric has the form

$$ ds^2 = -2\left(\frac{\partial^2}{\partial z^2} K\left(t,x,y,z\right)\right)d_t d_v + 2\left(\frac{\partial^2}{\partial z\partial y} K\left(t,x,y,z\right)\right)d_t d_w + d_t d_x + $$
$$ +2\left(\frac{\partial^2}{\partial z\partial x} K\left(t,x,y,z\right)\right)d_v d_w + d_v d_y - 2\left(\frac{\partial^2}{\partial y\partial x} K\left(t,x,y,z\right)\right){d_w}^2 + d_w d_z. $$

As example the metric is the Ricci-flat for the flows of the form

$$ U\left(t,x,y,z\right) = -1/2\,x\frac{\partial}{\partial z}F\left(t,z\right), \quad V\left(t,x,y,z\right) = -1/2\,y\frac{\partial}{\partial z}F\left(t,z\right), \quad W\left(t,x,y,z\right) = F\left(t,z\right), $$

with the function $F\left(t,z\right)$ which is determined from the equation

$$ -\mu\frac{\partial^3}{\partial z^3} + F\left(t,z\right)\frac{\partial^3}{\partial z^3}F\left(t,z\right) + \frac{\partial^2}{\partial z\partial t}F\left(t,z\right) = 0, $$

for which

$$ B(t,\vec{x}) = \frac{\partial}{\partial t}F\left(t,z\right) + F\left(t,z\right)\frac{\partial}{\partial z}F\left(t,z\right) $$
$$ H(t,\vec{x}) = -1/2x\frac{\partial^2}{\partial z\partial t}F\left(t,z\right) + 1/4x\frac{\partial}{\partial z}F\left(t,z\right)^2 - $$

$$- 1/2xF\left(t,z\right)\frac{\partial^2}{\partial z\partial z}F\left(t,z\right)+1/2\,\mu x\frac{\partial^3}{\partial z\partial z\partial z}F\left(t,z\right),$$

$$E(t,\vec{x}) = -1/2y\frac{\partial^2}{\partial z\partial t}F\left(t,z\right)+1/4y\frac{\partial}{\partial z}F\left(t,z\right)^2 - $$

$$- 1/2yF\left(t,z\right)\frac{\partial^2}{\partial z\partial z}F\left(t,z\right)+1/2\,\mu y\frac{\partial^3}{\partial z\partial z\partial z}F\left(t,z\right).$$

The metric of the form

Theorem 2.

$$ds^2 = 2\,dx^2 + 2\,dxdy + 2\,dxdu + 2\,dy^2 + 2\,dydz + 2\,dydv + 2\,dz^2 + 2\,dzdw + \\ + 2\,dtdp + 2\,d\eta d\xi + 2\,d\rho d\chi + 2\,dmdn + A dt^2 + B d\eta^2 + C d\rho^2 + E dm^2, \tag{1}$$

where

$$A = 2 - U\left(x,y,z,t\right)u - V\left(x,y,z,t\right)v - W\left(x,y,z,t\right)w,$$

$$B = \left(-UW + \mu\frac{\partial}{\partial z}U\right)w + \left(-UV + \mu\frac{\partial}{\partial y}U\right)v + \left(\mu\frac{\partial}{\partial x}U - (U)^2 - P\right)u - Up,$$

$$C = \left(-VW + \mu\frac{\partial}{\partial z}V\right)w + \left(\mu\frac{\partial}{\partial y}V - (V)^2 - P\right)v + \left(-UV + \mu\frac{\partial}{\partial x}V\right)u - Vp,$$

$$E = \left(-\mu\frac{\partial}{\partial x}U - \mu\frac{\partial}{\partial y}V - (W)^2 - P\right)w + \left(-VW + \mu\frac{\partial}{\partial y}W\right)v + \left(-UW + \mu\frac{\partial}{\partial x}W\right)u - Wp$$

is the Ricci-flat on solutions of Navier-Stokes system of equations.

Geometric characteristics of the 6D- metric depend on choice of the functions $H(t,\vec{x}), B(t,\vec{x}), E(t,\vec{x})$ and considerations of their properties joint with the metric of 14D-space can be used to determination of the functions

$$U(t,\vec{x}), \qquad V(t,\vec{x}), \qquad W(t,\vec{x}), \qquad P(t,\vec{x})$$

from the NS-equations:

$$\frac{\partial}{\partial t}U\left(t,\vec{x}\right) + U\left(t,\vec{x}\right)\frac{\partial}{\partial x}U\left(t,\vec{x}\right) + V\left(t,\vec{x}\right)\frac{\partial}{\partial y}U\left(t,\vec{x}\right) + W\left(t,\vec{x}\right)\frac{\partial}{\partial z}U\left(t,\vec{x}\right) -$$

$$-\mu\left(\frac{\partial^2}{\partial x^2}U\left(t,\vec{x}\right) + \frac{\partial^2}{\partial y^2}U\left(t,\vec{x}\right) + \frac{\partial^2}{\partial z^2}U\left(t,\vec{x}\right)\right) + H\left(t,\vec{x}\right) = 0,$$

$$\frac{\partial}{\partial t}V\left(t,\vec{x}\right) + U\left(t,\vec{x}\right)\frac{\partial}{\partial x}V\left(t,\vec{x}\right) + V\left(t,\vec{x}\right)\frac{\partial}{\partial y}V\left(t,\vec{x}\right) + W\left(t,\vec{x}\right)\frac{\partial}{\partial z}V\left(t,\vec{x}\right) -$$

$$-\mu\left(\frac{\partial^2}{\partial x^2}V\left(t,\vec{x}\right) + \frac{\partial^2}{\partial y^2}V\left(t,\vec{x}\right) + \frac{\partial^2}{\partial z^2}V\left(t,\vec{x}\right)\right) + E\left(t,\vec{x}\right) = 0,$$

$$\frac{\partial}{\partial t}W\left(t,\vec{x}\right) + U\left(t,\vec{x}\right))\frac{\partial}{\partial x}W\left(t,\vec{x}\right) + V\left(t,\vec{x}\right)\frac{\partial}{\partial y}W\left(t,\vec{x}\right) + W\left(t,\vec{x}\right)\frac{\partial}{\partial z}W\left(t,\vec{x}\right) -$$

$$-\mu\left(\frac{\partial^2}{\partial x^2}W\left(t,\vec{x}\right) + \frac{\partial^2}{\partial y^2}W\left(t,\vec{x}\right) + \frac{\partial^2}{\partial z^2}W\left(t,\vec{x}\right)\right) + B\left(t,\vec{x}\right) = 0$$

$$\frac{\partial}{\partial x}U\left(t,\vec{x}\right) + \frac{\partial}{\partial y}V\left(t,vecx\right) + \frac{\partial}{\partial z}W\left(t,x,y,z\right) = 0.$$

References

[1] Dryuma V. *On spaces related t the Navier-Stokes equations.* Bul. Acad. Ştiinţe Repub. Moldova, Mat., 2010, No. 3(64), 107–110.

[2] Dryuma V. *The Ricci-flat spaces related t the Navier-Stokes equations.* Bul. Acad. Ştiinţe Repub. Moldova, Mat., 2012, No. 2(69), 99–102.