On the construction and classification of the common invariant solutions for the $(1+3)$-dimensional Euler-Lagrange-Born-Infeld and homogeneous Monge-Ampere equations

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A solution of many problems of the theory of minimal surfaces, nonlinear electrodynamics, geometric optics, theories of gravity, geometry, unified field theory, string theories, black holes, cosmology, etc. is reduced to the investigation of the Euler-Lagrange equations [1, 2, 3, 4, 5, 6, 7], the Born-Infeld equations [8, 9, 10, 11, 12, 13], the Monge-Ampère equations [14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27] in the spaces of different dimensions and different types.

We consider the following $(1+3)$-dimensional PDEs:

- the Euler-Lagrange-Born-Infeld equation

$$\Box u\,(1-u_\nu u^\nu)+u^\mu u^\nu u_{\mu\nu}=0,$$

- the homogeneous Monge-Ampère equation

$$\det\left(u_{\mu\nu}\right)=0,$$

where $u=u(x),\ x=(x_0,x_1,x_2,x_3)\in M(1,3),\ u_\mu\equiv\dfrac{\partial u}{\partial x^\mu},\ u_{\mu\nu}\equiv\dfrac{\partial^2 u}{\partial x^\mu\partial x^\nu},\ u^\mu=g^{\mu\nu}u_\nu,\ g_{\mu\nu}=(1,-1,-1,-1)\delta_{\mu\nu},\ \mu,\nu=0,1,2,3$, and $\Box$ is the d'Alembert operator.

Here, $M(1,3)$ is a four-dimensional Minkowski space.

From the results obtained by W.I. Fushchich, W.M. Shtelen and N.I. Serov [28] it follows, in particular, that the common symmetry group of those equations is the generalized Poincaré group $P(1,4)$.

At the present time, we have constructed as well as classified some common invariant solutions of the equations under study. To obtain those results, we have used the results of the classification of symmetry reductions and invariant solutions of the Euler–Lagrange–Born–Infeld equation [29, 30].

In our report, I plan to present some of the results obtained.

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