Normal forms of functions with degenerate critical points on surfaces whose stabilizers are homotopically non-trivial

Bohdan Feshchenko
(Institute of Mathematics of NAS of Ukraine, Kyiv, Ukraine)
E-mail: fb@imath.kiv.ua

Let M be a smooth compact surface, and P be either $\mathbb{R}$ or S^1. The group of diffeomorphisms $\mathcal{D}(M)$ acts on the space of smooth P-valued functions $C^\infty(M,P)$ by the rule:

$$C^\infty(M,P) \times \mathcal{D}(M) \to C^\infty(M,P) \qquad (f,h) \mapsto f \circ h.$$

For a smooth function $f \in C^\infty(M,P)$ we denote by

$$\mathcal{S}(f) = \{h \in \mathcal{D}(M) \,|\, f \circ h = f\}, \qquad \mathcal{O}(f) = \{f \circ h \,|\, h \in \mathcal{D}(M)\}$$

the *stabilizer* and the *orbit* of f. Homotopy properties of $\mathcal{S}(f)$ and $\mathcal{O}(f)$ and their connected components are well studied for a large class of smooth functions with isolated singularities on surfaces, see [2]. We also denote by $\mathcal{S}_{\rm id}(f)$ a connected component of the identity map id in $\mathcal{S}(f)$.

We consider the following class $\mathcal{F}(M,P)$ of smooth functions: a function f belongs to $\mathcal{F}(M,P)$ if

(1) for each connected component V of the boundary ∂M a function $f|_V$ either takes a constant value or is a covering map,
(2) a set of critical points Σ_f of f is a disjoint union of smooth submanifolds of M and $\Sigma_f \subset \mathrm{Int}(M)$,
(3) for each connected component C of Σ_f and each critical point $p \in C$ there exist a local chart $(U, \phi : U \to \mathbb{R}^2)$ near p and a chart $(V, \psi : V \to \mathbb{R})$ near $f(p) \in P$ such that $f(U) \subset V$ and a local representation $f_p = \psi \circ f \circ \phi^{-1} : \phi(U) \to \psi(V)$ of f is
   (a) either a polynomial homogeneous polynomial f_p without multiple factors,
   (b) or is given by $f_C(x,y) = \pm y^{n_C}$ for some $n_C \in \mathbb{N}_{\geq 2}$ depending of C.

Connected components of Σ_f are isolated critical points and critical circles.

Let $\mathcal{F}^0(M,P)$ be a subset of $\mathcal{F}(M,P)$ of function which satisfy (1), (2), (3.b), but instead (3.a) the following condition holds:

(3.a') either a polynomial f_p given by $f_p(x,y) = \pm x^2 \pm y^2$.

For a function $f \in \mathcal{F}(M,P)$ a stabilizer $\mathcal{S}_{\rm id}(f)$ is homotopy equivalent to S^1 if $f \in \mathcal{F}^0(M,P)$, and is contractible otherwise, [1, Theorem 1.2]. Our main result is an analytical characterization of functions from $\mathcal{F}^0(M,P)$, see Theorem 3. The following proposition contains basic facts about functions from $\mathcal{F}^0(M,P)$.

Proposition 1. *Let f be a function from $\mathcal{F}^0(M,P)$. Then*

(1) *M is one of the following surfaces: $S^1 \times [0,1]$, D^2, S^2 or T^2.*
(2) *a function $f : M \to P$ is always null-homotopic if M is not a torus. A function on torus can be either null-homotopic or not null-homotopic.*
(3) *f has any finite number of critical circles if $M = S^1 \times [0,1]$, D^2, S^1, or T^2 and f is not null-homotopic. If $M = T^2$ and f is null-homotopic, then f has at least 2 critical circles.*
(4) *If $M = S^1 \times [0,1]$ or T^2, then f does not have isolated critical points. If $M = D^2$ or S^2, a function f has one and two non-degenerate extremes respectively.*

To state our main result we need the following definition.

Definition 2 (Primitive functions)**.** Let $f_0 : M \to P$ be a smooth function

(1) $M = S^1 \times [0,1] = \{(z,s) \,|\, z \in \mathbb{C}, |z| = 1,\ 0 \leq s \leq 1\}$, and $f_0 : S^1 \times [0,1] \to \mathbb{R}$ is given by $f_0(\phi, s) = s$;

(2) $M = D^2 = \{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 \leq 1\}$, and $f_0 : D^2 \to \mathbb{R}$ is given by $f_0(x, y) = \pm x^2 \pm y^2$;
(3) $M = S^2 = \{(x, y, z) \,|\, x^2 + y^2 + z^2 = 1\}$, and $f_0(x, y, z) : S^2 \to \mathbb{R}$ is given by $f_0(x, y, z) = z$;
(4) $M = T^2 = \left\{(x, y, z) \in \mathbb{R}^3 \,\middle|\, \left(\sqrt{x^2 + y^2} - 2\right)^2 + z^2 = 1\right\}$, and $f_0 : T^2 \to \mathbb{R}$ is given by $f_0(x, y, z) = z$;
(5) $M = T^2 = \{(w, z) \in \mathbb{C}^2 \,|\, |z| = |w| = 1\}$, and $f_0 : T^2 \to S^1$ is given by $f_0(w, z) = z$.

Obviously that functions from (1)–(4) belongs to $\mathcal{F}^0(M, P)$. They are height functions for (1)–(4), and a function from (5) is an angular projection. These functions have a minimum possible number of critical submanifolds, and we will call them *primitive functions*.

Theorem 3. *Let f be a smooth function from $\mathcal{F}^0(M, P)$ and $f_0 \in \mathcal{F}^0(M, P)$ be a primitive function. A function f admits a decomposition*

$$f = \varkappa \circ f_0 \circ h^{-1} \tag{1}$$

for some diffeomorphism $h : M \to M$ and a smooth function $\varkappa : \mathrm{Im}(f_0) \to P$ satisfying the following conditions:

(A) *$\varkappa$ has the only finite number of critical points in which it is not flat, i.e., not all derivatives of $\varkappa$ at each critical point vanish,*
(B) *$\varkappa$ does not have extremes at $f_0(\Sigma^0_{f_0})$ and $f_0(\partial M)$.*

In particular, if $f \in \mathcal{F}^0(T^2, P)$ is null-homotopic, then f_0 is given by (4), and by (5) otherwise. A factorization (1) is not unique and depends on the choice of h.

References

[1] Bohdan Feshchenko. Homotopy type of stabilizers of circle-valued functions with non-isolated singularities on surfaces, arXiv:2305.08255, 9 p., 2023
[2] Sergiy Maksymenko. Deformations of functions on surfaces *Proceedings of Institute of Mathematics of NAS of Ukraine*, 17, no. 2 (2020) 150-199.