Riemann Integration on a space with a fractal structure

José F. Gálvez-Rodríguez
(University of Almería)
E-mail: jgr409@ual.es

Cristina Martín-Aguado
(University of Almería)
E-mail: cristina_martinaguado@yahoo.es

Miguel A. Sánchez-Granero
(University of Almería)
E-mail: misanche@ual.es

In this work we start developing a Riemann-type integration theory on spaces which are equipped with a fractal structure (see [1] for more details). The definition of a fractal structure is the next one:

Definition 1. A fractal structure $\mathbf{\Gamma}$ on a set X is a countable family of coverings $\mathbf{\Gamma} = \{\Gamma_n : n \in \mathbb{N}\}$ such that Γ_{n+1} is a strong refinement of Γ_n for each $n \in \mathbb{N}$. Γ_2 is said to be a strong refinement of Γ_1 if Γ_2 is a refinement of Γ_1 (that is, each element of Γ_2 is contained in some element of Γ_1) and for each $B \in \Gamma_1$ it holds that $B = \bigcup\{A \in \Gamma_2 : A \subseteq B\}$. Cover Γ_n is called level n of the fractal structure.

We require to define a concept first:

Definition 2. Let $(X, \mathcal{S}, \mu)$ be a measure space and $\mathbf{\Gamma}$ be a fractal structure on X. $\mathbf{\Gamma}$ is said to be μ-disjoint if the following conditions hold:

(1) $\Gamma_n \subseteq \mathcal{S}$ is countable for each $n \in \mathbb{N}$.
(2) $\mu(B \cap J) = 0$ for each $B, J \in \Gamma_n$ such that $B \neq J$ and each $n \in \mathbb{N}$.
(3) $\mu(A) < \infty$ for each $A \in \Gamma_n$ and each $n \in \mathbb{N}$.

Next, we define the Darboux sums with respect to a measure and a fractal structure:

Definition 3. Let $(X, \mathcal{S}, \mu)$ be a measure space, $\mathbf{\Gamma} = \{\Gamma_n : n \in \mathbb{N}\}$ be a μ-disjoint fractal structure, and $f : X \to \mathbb{R}$ be a bounded function. Then, for each $J \in \Gamma_n$, we set $m(f; J) = \inf\{f(x) : x \in J\}$ and $M(f; J) = \sup\{f(x) : x \in J\}$, so that the lower and upper Darboux sums with respect to μ for each level of the fractal structure are given by

$$L(f; \Gamma_n, \mu) = \sum_{J \in \Gamma_n} m(f; J)\mu(J) \qquad \text{and} \qquad U(f; \Gamma_n, \mu) = \sum_{J \in \Gamma_n} M(f; J)\mu(J).$$

The lower and upper Riemann integrals with respect to a measure and a fractal structure are defined as follows:

Definition 4. Let $(X, \mathcal{S}, \mu)$ be a measure space, $\mathbf{\Gamma} = \{\Gamma_n : n \in \mathbb{N}\}$ be a μ-disjoint fractal structure on X, and $f : X \to \mathbb{R}$ be a bounded function. We define the lower and upper Riemann integrals of f with respect to μ and $\mathbf{\Gamma}$ on X as follows:

(1) Upper Riemann integral of f with respect to μ and $\mathbf{\Gamma}$:

$$\overline{\int_X}^{(\mu,\mathbf{\Gamma})} f := \inf\{U(f; \Gamma_n; \mu) : n \in \mathbb{N}\} = \lim_n U(f; \Gamma_n; \mu).$$

(2) Lower Riemann integral of f with respect to μ and $\mathbf{\Gamma}$:

$$\underline{\int_X}^{(\mu,\mathbf{\Gamma})} f := \sup\{L(f; \Gamma_n; \mu) : n \in \mathbb{N}\} = \lim_n L(f; \Gamma_n; \mu).$$

Now we give the defition of a Riemann-integrable function.

Definition 5. Let $(X, \mathcal{S}, \mu)$ be a measure space, $\mathbf{\Gamma} = \{\Gamma_n : n \in \mathbb{N}\}$ be a μ-disjoint fractal structure on X and $f : X \to \mathbb{R}$ be a bounded function. f is said to be Riemann-integrable with respect to μ and $\mathbf{\Gamma}$ on X if $\overline{\int_X}^{(\mu,\mathbf{\Gamma})} f$ is finite and $\underline{\int_X}^{(\mu,\mathbf{\Gamma})} f = \overline{\int_X}^{(\mu,\mathbf{\Gamma})} f$.

If f is Riemann-integrable with respect to μ and $\mathbf{\Gamma}$ on X, we define the Riemann integral of f with respect to μ and $\mathbf{\Gamma}$ on X, $\int_X^{(\mu,\mathbf{\Gamma})} f$, by $\int_X^{(\mu,\mathbf{\Gamma})} f = \underline{\int_X}^{(\mu,\mathbf{\Gamma})} f = \overline{\int_X}^{(\mu,\mathbf{\Gamma})} f$. We denote by $R(X; \mu; \mathbf{\Gamma})$ the set of Riemann-integrable functions with respect to μ and $\mathbf{\Gamma}$ on X.

The next step is defining the Riemann sum relative to a collection of points in a certain level of $\mathbf{\Gamma}$.

Definition 6. Let $\mathbf{\Gamma}$ be a fractal structure on a space X such that Γ_n is countable for each $n \in \mathbb{N}$. A selection for Γ_n is a collection of points $\xi := (x_A)_{A \in \Gamma_n}$ such that $x_A \in A$ for each $A \in \Gamma_n$.

Definition 7. Let $(X, \mathcal{S}, \mu)$ be a measure space, $\mathbf{\Gamma} = \{\Gamma_n : n \in \mathbb{N}\}$ be a μ-disjoint fractal structure on X and $f : X \to \mathbb{R}$ be a bounded function. Let $n \in \mathbb{N}$ and $\xi = (x_A)_{A \in \Gamma_n}$ be a selection for Γ_n. The Riemann sum for f relative to Γ_n, ξ and μ is defined as $S(f; \Gamma_n; \xi; \mu) := \sum_{A \in \Gamma_n} f(x_A)\mu(A)$.

The following theorem is analogous to the Riemann's Theorem in $\mathbb{R}^n$, but for bounded functions defined on a space with a μ-disjoint fractal structure.

Theorem 8. *Let $(X, \mathcal{S}, \mu)$ be a measure space, $\mathbf{\Gamma} = \{\Gamma_n : n \in \mathbb{N}\}$ be a μ-disjoint fractal structure on X, $f : X \to \mathbb{R}$ be a bounded function and $C \in \mathbb{R}$. The following statements are equivalent:*

(1) $f \in R(X; \mu; \mathbf{\Gamma})$ *and* $\int_X^{(\mu,\mathbf{\Gamma})} f = C$.
(2) Given $\varepsilon > 0$, *there exists* $n_0 \in \mathbb{N}$ *such that* $|C - S(f; \Gamma_n; \xi_n; \mu)| < \varepsilon$ *for each* $n \geq n_0$ *and each selection for* Γ_n, ξ_n.
(3) Given $\varepsilon > 0$, *there exists* $n \in \mathbb{N}$ *such that* $|C - S(f; \Gamma_n; \xi; \mu)| < \varepsilon$ *for each selection for* Γ_n, ξ.
(4) $S(f; \Gamma_m; \xi_m; \mu) \overset{m\to\infty}{\longrightarrow} C$ *for each sequence* (ξ_m) *such that* ξ_m *is a selection for* Γ_m *for each* $m \in \mathbb{N}$.

The next result is crucial in order to justify that the Riemann integral of a bounded function with respect to a measure and a fractal structure does not depend on the fractal structure.

Proposition 9. *Let $(X, \mathcal{S}, \mu)$ be a measure space, $\mathbf{\Gamma} = \{\Gamma_n : n \in \mathbb{N}\}$ and $\mathbf{\Gamma}^* = \{\Gamma_n^* : n \in \mathbb{N}\}$ be two μ-disjoint fractal structures on X and $f : X \to \mathbb{R}$ be a bounded function. If $f \in R(X; \mu; \mathbf{\Gamma})$ and $f \in R(X; \mu; \mathbf{\Gamma}^*)$, then $\int_X^{(\mu,\mathbf{\Gamma})} f = \int_X^{(\mu,\mathbf{\Gamma}^*)} f$.*

Hence, it does make sense to introduce the following concept:

Definition 10. Let $(X, \mathcal{S}, \mu)$ be a measure space and $f : X \to \mathbb{R}$ be a bounded function. f is said to be μ-Riemann-integrable if there exists a μ-disjoint fractal structure $\mathbf{\Gamma}$ on X such that f is Riemann-integrable on X with respect to μ and $\mathbf{\Gamma}$. Moreover, if so, the integral is defined as $\int_X^{\mu} f = \int_X^{(\mu,\mathbf{\Gamma})} f$.

Proposition 11. *Let $(X, \mathcal{S}, \mu)$ be a finite measure space and $f : X \to \mathbb{R}$ be a bounded measurable function. Then $f \in R(X; \mu)$ and $\int_X^{\mu} f = \int f d\mu$.*

Hence, if $\mathbf{\Gamma}$ is a μ-disjoint fractal structure on X such that f is Riemann-integrable with respect to μ and $\mathbf{\Gamma}$, we can calculate $\int f d\mu$ as $\int_X^{(\mu,\mathbf{\Gamma})} f$. It also follows that if μ is a finite measure on the Borel σ-algebra of a topological space X and $f : X \to \mathbb{R}$ is a bounded continuous map, then f is μ-Riemann-integrable on X.

References

[1] José F. Gálvez-Rodríguez, Cristina Martín-Aguado and Miguel A. Sánchez-Granero. Riemann Integral on Fractal Structures. *Mathematics*, 12(2) : 310, 2024.